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SUPERNOVAE, NEUTRINOS, AND NEUTRON STARS

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SUMMARY

The final evolution of a pre-supernova star is reviewed. It is now believed that the supernova explosion of a star is first triggered by the collapse of its core, followed by the sudden release of gravitational energy or the sudden ignition of nuclear fuel in the envelope. Among the subjects discussed herein are the causes for stellar collapse, the static structures of white dwarfs and neutron stars, the dynamic collapse of oversized neutron stars, angular momentum and stellar collapse, and the observable features of neutron stars.

CONTENTS

SUPERNOVAE, NEUTRINOS, AND NEUTRON STARS*

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1. INTRODUCTION‡

It is now accepted with reasonable assurance that the supernova explosion of a star is first triggered by the collapse of its core,** followed by the sudden release of gravitational energy or the sudden ignition of nuclear fuel in the envelope (References 2-6). A number of suggestions have been made as to the physical processes which may cause the core to collapse suddenly. Among the most promising mechanisms are the disintegration of iron (the then most abundant element) into helium at a temperature of 7×10^{9}°K (Reference 3), and the neutrino process which dissipates stellar energy at a drastic rate of 10^{23} ergs/cm^3-sec at the same temperature (this rate is to be compared with the stellar energy density which is 10^{24} ergs/cm^3 at the same temperature and a density of 10^6 gm/cm^3) (Reference 5).

As we shall see later, although these two physical processes give rise to different kinds of instabilities, nevertheless they are capable individually of triggering the sudden release of either gravitational or nuclear energy of a star. A detailed computation is underway to determine the range of masses for which either of these two processes comes into play first.***

A sudden release of nuclear or gravitational energy in the envelope will cause it to expand quickly (References 3 and 4). The expanding envelope is directly observable. Owing to its complexity and its dynamical nature, only qualitative conclusions have been presented. Even so, interesting results regarding the origin of cosmic rays (Reference 7) and the origin of elements have been obtained, and qualitative agreements with observational results are quite satisfactory (References 3 and 6).

What is left as the core collapses is not immediately clear. In the past, it was usually assumed that ordinary white dwarfs (whose mean density is around 10^6 gm/cm^3) are remnants of supernova explosions. The large number of white dwarfs discovered (around 10 percent of the stars in the solar neighborhood are white dwarfs), however, cannot be explained by the relative scarcity of

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†Goddard Institute for Space Studies and the Department of Physics, Columbia University. Mr. Chiu is a NRC-NAS Senior Research Associate and also a Research Fellow, Academia Sinica, Republic of China.
‡A more complete description of stellar evolution along the present trend of thought may be found in Reference 1.
**The concept of stellar collapse was first introduced by G. Gamow and M. Schönberg (Reference 2).
***H. Y. Chiu and E. E. Salpeter, work in progress.

supernovae per galaxy (which occur at a rate of around 1/50 – 300 years) (References 8 and 9). The other alternative that neutron stars may be the remnants of supernovae has so far been accepted only with skepticism* (References 10 and 11). Moreover, there is *no astronomical evidence* yet that such stars even exist.

Theoretically there are also some difficulties. At least one type of pre-supernova stars is believed to have a mass of around $30 M_{\odot}$ ($M_{\odot}$ = solar mass = 2×10^{33} gm) of which around 20 $M_{\odot}$ belongs to the core. Since there exists an upper limit ($\approx 0.8\ M_{\odot}$) for the mass of a neutron star (and also white dwarfs) beyond which no stable configuration exists, what will be the fate of the massive stellar core? It is unlikely that a large fraction of the 20 $M_{\odot}$ which the core possesses will be blown away by any reasonable means. Since the composition of the core is believed to be mainly iron – the end product of thermonuclear evolution, no nuclear energy is available and gravitational energy is incapable of doing so (Section 3.1).

It was also suggested that the collapse would initiate, in addition to an outward shock which ejects matter, also an inward shock wave which compresses the core quickly (Reference 4). When the density reaches 10^{14} gm/cm^3, the inter-nucleon distance will be of the order of the size of the repulsive core ($\approx 10^{-14}$ cm), and the shock wave will be rebounded by the hard nuclear core. The rebounded shock wave will expand the star to the normal density possessed by a white dwarf, and will eject more mass from the star. This hope, however, was relinquished recently because of neutrino processes. Neutrino processes involving electron neutrinos and μ-neutrinos are extremely efficient in dissipating energy – the relaxation time being around 10^{-5} sec at $T = 10^{12}$°K. This would be the temperature of a medium if it is compressed adiabatically to a density of 10^{13} gm/cm^3. This relaxation time (10^{-5} sec) is extremely short as compared with the time the compression takes place, which is calculated to be around 10^{-2} sec. Hence, once compressed to such high densities ($\gtrsim 10^{13}$ gm/cm^3), the star will lose all its energy; and it cannot restore itself to a normal density. This is discussed in more detail in Section 3.4.

So far, no stars in any way resembling neutron stars have been discovered. We would like to point out that this is not evidence against the existence of neutron stars. Our atmosphere strongly absorbs electromagnetic radiation whose wavelengths are less than 3000A. The total energy output by a neutron star is proportional to the fourth power of its effective surface temperature T_e. The part of neutron star radiation that will pass through our atmosphere, however, increases only as T, when $T \gg 10^4$°K (Section 8.3). For a surface temperature of 10^6°K, the total energy output of a neutron star is around that of the sun, but the part of radiation that can pass through our atmosphere is about 10^{-5} of that of the sun at the same distance. This is too faint to be seen at even a reasonable distance (say, 10 light years).

We shall discuss the nature of various problems related to the formation and detection of neutron stars. Although in the past, several pieces of excellent and wonderful work have been done,† the problem in the large is still unsolved. In this paper we do not attempt to solve

*However, W. Baade and F. Zwicky believed that supernova explosions are resulted from the formation of a neutron star (Reference 12). This belief is no longer supported by other evidences; see Reference 3.

†These works are summarized in Section 5.6. References to these works may also be found there.

this problem. We should like, however, to emphasize the difficulty of the problem and the possible direction of a solution.

Since rotational energy may be quickly dissipated by gravitational radiation when the collapse takes place, we shall ignore rotational effects in our discussions. This is discussed in particular detail in Section 7.

2. CAUSES FOR COLLAPSE OF A STAR

A star is, in the large, a mechanical system in quasi-hydrostatic equilibrium. It has a negative total energy and, in order to be in a stable equilibrium, its total energy must also be a minimum on its energy surface with respect to small perturbations. Moreover, the energetics of a stable star must be such that hydrostatic equilibrium has a useful meaning. When either of these two conditions is not fulfilled, the star will be unstable and, since its total energy is negative to start with, collapse will take place.

When the first condition is not fulfilled, the star is said to have *hydrostatic instability*. When the second condition is not fulfilled, the star is said to have *dynamical instability*. In the following we study them in more detail.

2.1. Hydrostatic Instability

Here we are concerned only with the stability of a star in hydrostatic equilibrium under small adiabatic perturbations (Reference 13).* Let E_T be the total energy of a star. E_T is given by

$$E_T = - G \int_0^M \frac{m\ dm}{r} + \int_0^M \epsilon\ dm \quad , \tag{2.1}$$

where the first term represents the gravitational energy and ϵ is the thermodynamic energy per unit mass, G is the gravitational constant, m is the mass of the star within a sphere of symmetry of radius r, and M is its total mass. In order that the star be in equilibrium, it is necessary that the first variation of E_T be null:

$$\delta E_T = 0 \ ; \tag{2.2}$$

and, in order that the equilibrium be a stable one, the sign of the second variation must be positive – that is,

$$\delta^2 E_T > 0 \quad . \tag{2.3}$$

*Also an unpublished paper by F. J. Dyson: "Hydrostatic Instability of a Star."

One can demonstrate that Equation 2.2 gives rise to the equation of hydrostatic equilibrium and Equation 2.3 gives rise to the following inequality:*

$$\int \left(\Gamma - \frac{4}{3}\right) P \, dV > 0 \ , \tag{2.4}$$

where P is the hydrostatic pressure and V is the volume of the star; Γ is the adiabatic exponent and is defined as

$$\Gamma = -\frac{v}{P}\left(\frac{dP}{dv}\right)_{ad} \tag{2.5}$$

The subscript means that this differentiation is carried out under an adiabatic process.

One form of Γ is† (see footnote, page 3)

$$\Gamma = \frac{Pv + v\left(\frac{\partial \epsilon}{\partial v}\right)_P}{P\left(\frac{\partial \epsilon}{\partial P}\right)_v} \ , \tag{2.6}$$

where v is the specific volume (volume per unit mass) and P is the gas pressure. Assuming that ϵ and P_v are all functions of temperature alone, and approximating Δv by v, one obtains the following form for Γ:

$$\Gamma = 1 + \frac{\Delta(Pv)}{\Delta\epsilon} \ . \tag{2.7}$$

Normally, for a non-relativistic gas, $Pv = 2/3\ \epsilon$; hence $\Gamma = 5/3$. For a relativistic gas, $Pv = 1/3\ \epsilon$; hence $\Gamma = 4/3$.‡ If, in a particular temperature regime phase transition takes place, the energy of transition has to be included in ϵ. Thus, when phase transition occurs (such as ionization dissociation or nuclear disintegration) the value of Γ will drop to below 4/3. For iron-helium transition, the heat of transition ϵ_t is roughly 2.2 Mev/nucleon. The transition occurs at a temperature of 8×10^9 °K, at which the thermal energy is around 1 Mev/particle. Since the electrons are quite relativistic, $Pv \approx 1/3\ \epsilon_e$, where ϵ_e is the thermal energy for the electrons. Using the value $\epsilon_e \approx 1$ Mev, and writing $\epsilon \approx \epsilon_e + \epsilon_t$ (thermal energy of nuclei and radiation energy may be neglected), we find that

$$\Gamma = 1 + \frac{\frac{1}{3}\epsilon_e}{\epsilon_e + \epsilon_t} \approx 1.1 \ . \tag{2.8}$$

*F. J. Dyson has presented a concise derivation of Equation 2.4 (see footnote, page 3), which is briefly described in Reference 5.
†Derivation of Equation 2.6 may be found in Reference 5.
‡Although Equation 2.7 is an approximate expression for Γ, in these two limiting cases it gives the correct value for Γ.

We remark that the dissociation of iron into helium is not the only mechanism that will cause Γ to drop to below 4/3. Figure 1 shows Γ as a function of temperature for a number of densities for a gas in equilibrium with electron pair creation (Reference 14).* One finds that the value of Γ may drop to below 4/3 if the density is low enough. For the density-temperature region one usually considers inside a star, the creation of electron-positron pairs will not cause a star to collapse. Only for extremely massive objects, which Fowler and Hoyle considered as possible radio sources, will the creation of electron-positron pairs be responsible for the collapse of a star (Reference 15).

The disintegration of iron into helium, however, occurs over an extensive (though narrow) temperature-density region as indicated in Figure 2. Hence, every star which evolves with increasing temperature will come across such a region. In these regions instabilities may occur. Although the sign of the integral in Equation 2.4 has not been proven to be generally negative for a star in this density-temperature region, in certain simple cases (References 4 and 5) it has negative values. A more extensive work to study this kind of instability is underway.†

2.2. Dynamical Instability of a Star

While a star is stable in a hydrostatic sense, it may still be unstable

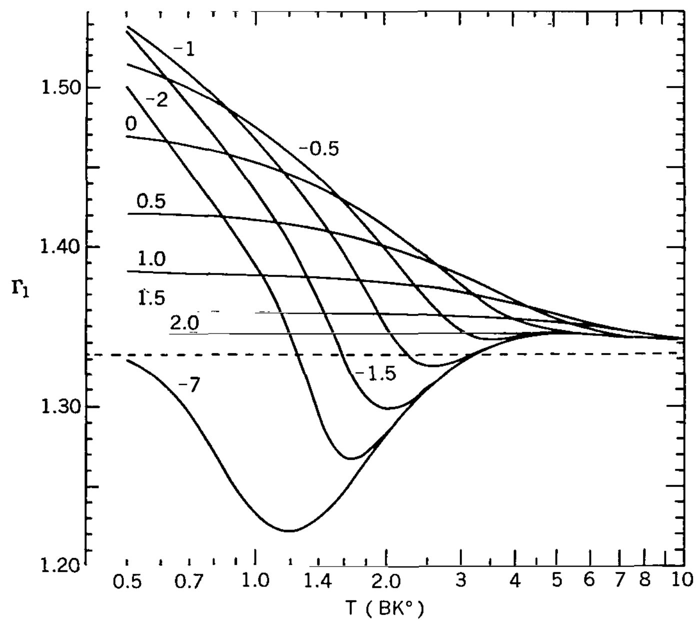

Figure 1—The adiabatic exponent $\Gamma \equiv -\frac{v}{P}\left(\frac{dP}{dv}\right)_{ad}$ for an electron gas in equilibrium with radiation and pair creation is plotted as a function of temperature for a number of densities. The numbers on the curves refer to $\log_{10}(\rho/\rho_o)$, where $\rho_o = 2 \times 10^6$ gm/cm³.

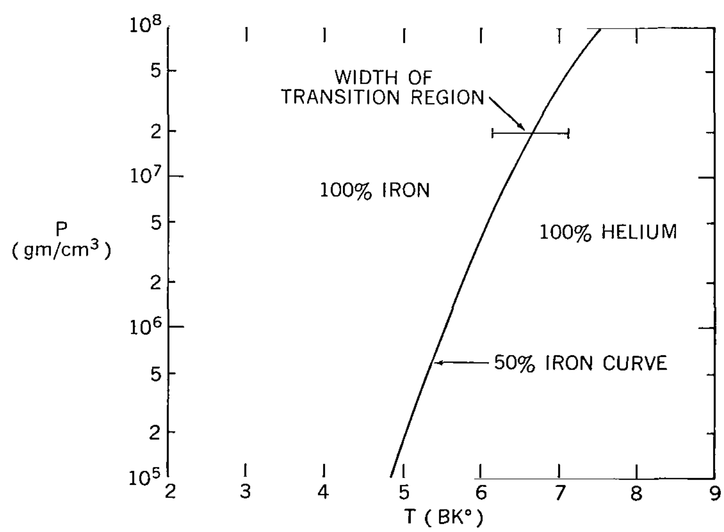

Figure 2—The temperature-density relation, for which the equilibrium ratio Fe^{56} to He^4 is unity, is plotted. The transition region has a width of around $\Delta T \approx 10^9$ °K.

*Also unpublished material by H. Y. Chiu and S. Tsuruta: "Thermodynamic Properties of Hot Matter."
†See footnote ***on, page 1.

dynamically, and vice versa. The dynamical stability of a star is related to the energy flow inside a star.

From the condition for hydrostatic equilibrium ($\delta E_T = 0$) one may obtain the equation for hydrostatic equilibrium:

$$\frac{dP}{dr} = -\rho \frac{Gm}{r^2}, \tag{2.9}$$

where ρ is the density and m was defined previously and is given by

$$m = \int_0^r 4\pi r^2 \rho \, dr \quad . \tag{2.10}$$

A more detailed discussion of Equations 2.9 and 2.10 may be found in standard textbooks (Reference 16). Equation 2.9 is valid only when all matter inside a star does not have macroscopic motion. When a star is contracting rapidly, one must include the acceleration $\dot{v}$ (where v is the velocity and $\dot{v}$ its hydrodynamic derivative). In the absence of relativistic effect, and assuming spherical symmetry, both v and $\dot{v}$ are in the direction of r (notation: v_r). Hence, the equation for hydrostatic equilibrium becomes

$$\frac{dP}{dr} = -\rho \frac{Gm}{r^2} - \rho \dot{v}_r \quad ; \tag{2.11}$$

$\dot{v}_r$ is given by

$$\dot{v}_r = \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} \quad . \tag{2.12}$$

It is more convenient to use m as one of the two independent variables since m is independent of t. Then Equation 2.11 becomes (we now drop the subscript r on v and $\dot{v}$)

$$\frac{dP}{dm} = -\frac{G}{4\pi} \frac{m}{r^4} - \frac{\dot{v}}{4\pi r^2} \quad . \tag{2.13}$$

Differentiating Equation 2.10, one obtains a relation between r and m:

$$\frac{dr}{dm} = \frac{1}{4\pi r^2 \rho} \quad . \tag{2.14}$$

Further, we also have the equation of continuity:

$$\dot{\rho} + \rho \, \mathrm{div}\, \mathbf{v} = \dot{\rho} + \rho \frac{\partial v}{\partial r} = 0 \quad . \tag{2.15}$$

The contraction of the star is caused by some fast energy dissipation process (like the neutrino process) which depends only on the density ρ and the temperature T. Let the rate of energy dissipation per unit mass be $\epsilon_d(\rho, T)$. The energy balance equation is

$$-T \frac{dS}{dt} = \epsilon_d(\rho, T) , \tag{2.16}$$

where S is the entropy per unit mass.* Further, thermodynamics supplies us with the following relations:

$$\left.\begin{aligned} S &= S(\rho, T) , \\ P &= P(\rho, T) . \end{aligned}\right\} \tag{2.17}$$

Hence, the four equations, Equations 2.13 through 2.16, with their four unknown variables (ρ, T, r, v), can now be solved in terms of m and t (which are the independent variables). The boundary conditions are:

$$\left.\begin{array}{lll} \text{At } m = 0: & & r = v = \dot{v} = 0, \ \rho \text{ and } T \text{ are finite, and } \frac{dP}{dm} \propto m^{-1/3}; \\ \text{At some value of } r: & & v, \dot{v} \text{ are finite; and } \rho \text{ and } T \text{ tend to zero simultaneously;} \\ \text{At } t = 0: & & \rho = \rho_o(m), \ T = T_o(m), \ v = v(m), \ \dot{v} = \dot{v}(m). \end{array}\right\} \tag{2.18}$$

These boundary conditions are physical restrictions on the type of solutions one expects from the set of nonlinear differential equations, Equations 2.13 through 2.16. Their meaning is almost self-evident. Equations 2.13 through 2.16 with the boundary condition Equation 2.18 describe the non-relativistic collapse of a star. If v approaches a certain limit which will be discussed below, it may happen that no solutions could be found which are continuous in both m and t and satisfy the boundary condition (Equation 2.18). In such cases a shock wave will be developed. The behavior of some simple shock wave solutions has been studied by Colgate el al. (Reference 4).

We now study the sign of $\dot{v}$ and the physical limit on v. Let W be the gravitational energy of the star and E its total thermodynamic energy; then the total energy of the star E_T is given by

$$E_T = W + E . \tag{2.19}$$

The virial theorem gives†

$$-W = (1 + \eta) E , \tag{2.20}$$

*This equation is derived on the assumption that the rate of energy transfer by photon is very small compared with the rate of energy dissipation by neutrinos. For the justification of this assumption, see Reference 5.
†See Appendix A.

where η is a slowly varying constant depending on the gross thermodynamic properties of a star. For all cases, $1 > \eta > 0$. At $T \approx 6 \times 10^9$ °K, $\eta \approx 0.1$.*

Approximately one can write

$$W = -G \int \frac{m\,dm}{r} \approx -\frac{GM^2}{R} , \tag{2.21}$$

where R is the radius of the star, and also

$$E = \int \epsilon \, dm \approx M \cdot R_g T , \tag{2.22}$$

where ϵ, the thermodynamic energy per unit mass, is approximated by the expression for a perfect gas:

$$\epsilon \approx R_g T \tag{2.23}$$

and R_g is the gas constant $\left(R_g = 8.2 \times 10^7 \text{ in cgs units}\right)$. From Equations 2.21 and 2.22, the following relations are obtained:

$$T = \frac{GM}{R_g(1+\eta)} \frac{1}{R} , \tag{2.24}$$

$$W = -\eta E_T = -\eta M R_g T = -\frac{\eta G}{1+\eta} \frac{M^2}{R} . \tag{2.25}$$

Since gravitational contraction is the only energy source available at $T \gtrsim 4 \times 10^9$ °K, above this temperature the energy balance equation becomes

$$-\frac{dW}{dt} = \int \epsilon_d \, dm \approx M \epsilon_d(\rho, T) . \tag{2.26}$$

The energy dissipation function ϵ_d for the annihilation process (which is the most important one in this temperature density region) is

$$\epsilon_d = \epsilon_o \frac{T^9}{\rho} , \tag{2.27}$$

where $\epsilon_o = 4 \times 10^{-75}$ ergs/(°K)^9-sec-cm^3.

*See Appendix A.

From Equation 2.24 we can also obtain a dynamical relation between T and ρ. Since

$$\rho = \frac{4}{3}\pi R^3 ,$$

the relation is

$$\rho = \left[\frac{Rg^3}{4G^3 M^2}\right] T^3 \equiv AT^3 . \tag{2.28}$$

A is a constant defined by the above equation.

From Equations 2.25 through 2.28, we obtain the following relations:

$$-\frac{dR}{dt} = \frac{1+\eta}{\eta GM} e_o \frac{1}{A}\left[\frac{GM}{R_g(1+\eta)}\right]^6 \frac{1}{R^4} \equiv \frac{C}{R^4} \tag{2.29}$$

and

$$-\frac{d^2R}{dt^2} = -\frac{4C}{R^5}\frac{dR}{dt} = +\frac{4C^2}{R^9} . \tag{2.30}$$

C is a constant depending on M and other gas characteristics. We now identify v with dR/dt and $\dot{v}$ with d^2R/dt^2. They both have unchanging negative signs which indicate that the star is collapsing inward, with rapidly increasing speed. Although Equations 2.29 and 2.30 are only approximate solutions, they do indicate strongly that a stellar collapse will eventually occur.

The limit for $\dot{v}$ is set by the gravitational field of the star. Thus,

$$\dot{v} < \frac{MG}{R^2} . \tag{2.31}$$

Equation 2.31, together with Equation 2.30, sets a lower limit on R below which no hydrostatic equilibrium is possible:

$$R > \left[\frac{4C^2}{MG}\right]^{1/7} \equiv R_l . \tag{2.32}$$

A similar result may be obtained by considering the upper limit for v. The upper limit for v is set by the gravitational energy available at radius R:

$$\frac{1}{2} Mv^2 < \frac{M^2G}{R} . \tag{2.33}$$

Equation 2.33 is identical with Equation 2.32 apart from a numerical factor close to unity.

Let $M = 20\ M_\odot$, $\eta = 0.1$; we find the limiting radius R_l to be

$$R_l \approx 10^9 \text{ cm} ,$$

$$C = 1.08 \times 10^{31} \text{ (in cgs units)} .$$

The corresponding temperature as estimated from Equation 2.28 is roughly 3×10^9 °K. This temperature is below the iron-helium disintegration temperature ($\approx 6 \times 10^9$ °K.) However, we remark that the above estimate is very crude. The only conclusion we can draw is that both the *annihilation neutrino process* and the *iron-helium disintegration process* are capable of causing a massive star to collapse. To answer this question more quantitatively, one needs to find out the sign of the integral (Equation 2.4) and to solve Equations 2.13 through 2.17 with the boundary condition Equation 2.18 and appropriate input parameters.

The steep dependence of $\dot{v}$ on $1/R$ ($\propto T$) as shown in Equation 2.30 indicates that the collapse may take place suddenly.

3. PHYSICAL EVENTS AND CONSEQUENCES OF STELLAR COLLAPSE

We have demonstrated in the last section that, when the gravitational contraction of a star (caused by neutrino emission process) reaches a certain stage, collapse must take place and the concept of hydrostatic equilibrium loses its meaning. We shall find out whether the collapse will stop and whether the star can eject enough mass so as to become an ordinary white dwarf.

3.1. Impossibility of Ejecting Enough Matter To Become a White Dwarf

In order that a massive core may eject enough mass to become a white dwarf, some physical mechanism must be present to eject almost all of its mass into space.

It is difficult to imagine a practical mechanism to achieve this. However, we shall now demonstrate on energetic grounds that a star cannot eject enough matter into space so that the remaining star becomes a white dwarf. The gravitational energy of a star is GM^2/R. Since no cold star of mass $> 1.4\ M_\odot$ may exist (see Section 4), an energy of the amount

$$\frac{G\left(M^2 - M_\odot^{\ 2}\right)}{R}$$

is needed to dissociate the star from its gravitational binding. This energy must be supplied gravitationally by the remaining core of the star of mass around $M_\odot$, radius r. Hence, to the first approximation

$$\frac{G\left(M^2 - M_\odot^{\ 2}\right)}{R} \approx \frac{GM_\odot^{\ 2}}{r} . \tag{3.1}$$

Consider $M \approx 20\,M_\odot$, and a radius of 10^9 cm (which corresponds to a temperature of around 1 Mev when instability occurs); we find $r \approx R/400$. The density of the remaining core will be $(400)^3 = 6.4 \times 10^7$ higher than that of the star before collapse takes place. From Equation 2.28 the density at the instant of collapse is estimated to be roughly 10^7 gm/cm^3. Hence the remaining core will have a density of at least 10^{13} gm/cm^3, which is the density of a neutron star.

This example is cited just to demonstrate that, when a massive star collapses, there is no possibility that the remaining core will be a white dwarf.

Moreover, most of the energy released during the gravitational collapse of a star will be in the form of neutrinos and cannot be of any practical use. This will be discussed in more detail in Section 3.4.

We now examine the thermodynamic properties of dense matter and the corresponding stars.

3.2. Thermodynamic Properties of Degenerate Matter at Near Zero Temperatures

At zero temperature and reasonably low density ($\rho \lesssim 10^{12}$ gm/cm^3) the energetics of an ionized gas are dominated by the electrons. The available energy states are occupied up to a Fermi momentum p_F corresponding to a Fermi energy E_F. The relation between the electron number density n_e and Fermi momentum p_F and density ρ are as follows:*

$$x \equiv \frac{p_F}{m_e c} , \tag{3.2}$$

$$n_e = \frac{8\pi m_e^3 c^3}{3h^3} x^3 = 5.9 \times 10^{29} x^3/\text{cm}^3 , \tag{3.3}$$

$$\rho = 10^6 \frac{A}{Z} x^3 \text{ gm/cm}^3 , \tag{3.4}$$

where A and Z are average values for the mass number and the atomic number for the gas. The electron Fermi energy E_F is given by

$$E_F = m_e c^2 \left[\left(1 + x^2 \right)^{1/2} - 1 \right] . \tag{3.5}$$

Salpeter (Reference 18) found that in the stellar regime the pressure energy density relation is very well described by that for a perfect Fermi gas. All other corrections (Coulomb

*For a description of the theory of a perfect Fermi gas, see Reference 17.

correction, exchange effect, ion motion, etc.) are very small. These relations for a perfect Fermi gas are (Reference 17):

$$P = \frac{\pi m^4 c^5}{3h^3} f(x) = 6.01 \times 10^{22} f(x) \text{ d/cm}^2 , \tag{3.6}$$

$$E_{kin} = \frac{\pi m^4 c^5}{3h^3} g(x) , \tag{3.7}$$

$$f(x) \equiv x(2x^2 - 3)(x^2 + 1)^{1/2} + 3 \sinh^{-1} x , \tag{3.8}$$

$$g(x) \equiv 8x^3\left[(x^2 + 1)^{1/2} - 1\right] - f(x) , \tag{3.9}$$

where E_{kin} is the kinetic energy density of the electrons. We have the following asymptotic expression:

$$\left.\begin{aligned} f(x) &\rightarrow \frac{8}{5} x^5 \\ g(x) &\rightarrow \frac{12}{5} x^5 \end{aligned}\right\} \qquad x \rightarrow 0 , \tag{3.10}$$

and

$$\left.\begin{aligned} f(x) &\rightarrow 2x^4 \\ g(x) &\rightarrow 6x^4 \end{aligned}\right\} \qquad x \rightarrow \infty . \tag{3.11}$$

Hence

$$P \rightarrow \frac{2}{3} E_{kin} \qquad x \rightarrow 0 , \tag{3.12}$$

$$P \rightarrow \frac{1}{3} E_{kin} \qquad x \rightarrow \infty , \tag{3.13}$$

and

$$P = \frac{1}{20}\left(\frac{3}{\pi}\right)^{2/3} \frac{h^2}{m} n_e^{5/3} \qquad x \rightarrow 0 , \tag{3.14}$$

$$P = \frac{1}{8}\left(\frac{3}{\pi}\right)^{1/3} hcn_e^{4/3} \qquad x \rightarrow \infty . \tag{3.15}$$

3.3. Composition of Degenerate Matter at Near Zero Temperatures

If the energy of an electron is greater than the ϵ_z, the beta-decay energy for electron emission (counting only the kinetic energy) of the nucleus (Z-1, A), then inverse beta reactions involving (Z, A) will occur:

$$e^- + (Z,A) \longrightarrow (Z-1,A) + \nu \quad . \tag{3.16}$$

If $E_F > \epsilon_z$, then the nucleus (Z, A) ceases to be stable and undergoes inverse beta decay to the nucleus (Z-1, A). The nucleus (Z-1, A) cannot decay because all electronic states are occupied up to the Fermi energy $E_F > \epsilon_z$. If the nuclei are in equilibrium, the composition is easily calculated. By minimizing the quantity

$$b^* = \frac{B(Z,A) - ZE_F'}{A} \tag{3.17}$$

(where E_F' is the Fermi energy of the electron gas minus the neutron-hydrogen atom rest mass energy difference), Salpeter (Reference 18) calculated the equilibrium composition for cold degenerate matter. His result is shown in Table 1.

When $\rho \gtrsim 10^7$, Γ for the electron gas is already very close to 4/3 (Figure 1). The gas sphere is only marginally stable against hydrostatic instability. The inverse beta reactions will cause Γ to be less than 4/3 for the density region $\rho = 10^7$ to $\rho = 10^{12}$ gm/cm^3. Wheeler (Reference 19) has computed the ratio of P to $\rho^{4/3}$; he finds that the ratio actually decreases with ρ in the above density region.* His result is shown in Figure 3.

If all nuclei are not in thermodynamic equilibrium, then the transition may not take place as described by

Table 1

Equilibrium Composition of Cold Matter*

(Z, A)	(26,56)	(28,62)	(28,64)	(28,66)	(28,68)
E_F	0.6	2.5	3.9	6.1	7.0
$\log_{10}\rho$	7.15	8.63	9.15	9.69	9.87
(Z, A)	(30,76)	(30,78)	(30,80)	(32,90)	(38,120)
E_F	8.5	9.5	14.8	20.6	24.0 Free neutrons
$\log_{10}\rho$	10.13	10.28	10.84	11.28	11.53

*ρ is measured in gm/cm^3, E_F in Mev.

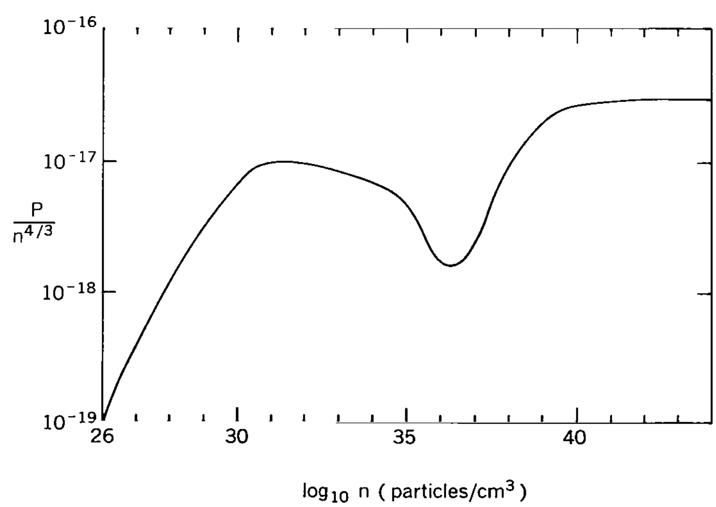

Figure 3—The ratio $P/n^{4/3}$ (where n is the total number density of nucleons) is plotted as a function of n. In between $n = 10^{31}/\text{cm}^3$ and $n = 10^{38}/\text{cm}^3$ ($\rho = 10^7 \rightarrow 1.67 \times 10^{13}$ gm/cm^3) $P/n^{4/3}$ decreases as n increases, indicating that the power dependence of P on n is less than 4/3; this result is due to Wheeler.

*At zero temperature $\Gamma = (\rho/P)(dP/d\rho)$. If P is expressed as $P = P_0\,\rho^\Gamma$, where P_0 is a proportionality constant, then Γ is the adiabatic exponent.

Table 1. Cameron (Reference 20) has studied the rate of nuclear reactions at high density (pycno nuclear reactions). He found that most nuclear reactions leading towards equilibrium state will take place spontaneously even at zero temperature when the density is greater than 10^9 gm/cm^3.

We may therefore safely expect that the composition of matter at $\rho > 10^9$ gm/cm^3 will be reasonably given by Table 1 under equilibrium conditions. Wheeler's result (Figure 3) indicates that in-between the density-temperature region $\rho = 10^8$ to $\rho = 10^{12}$ gm/cm^3 there are no equilibrium configurations for a star. Harrison, Wakano, and Wheeler (Reference 19) have also studied the structure of stars at zero temperature; the result is shown in Figure 4, in which the mass as a function of stellar density is plotted. In certain density regions where inverse beta reaction occurs, the mass of the star decreases with increasing central density, indicating that such configurations are unstable.

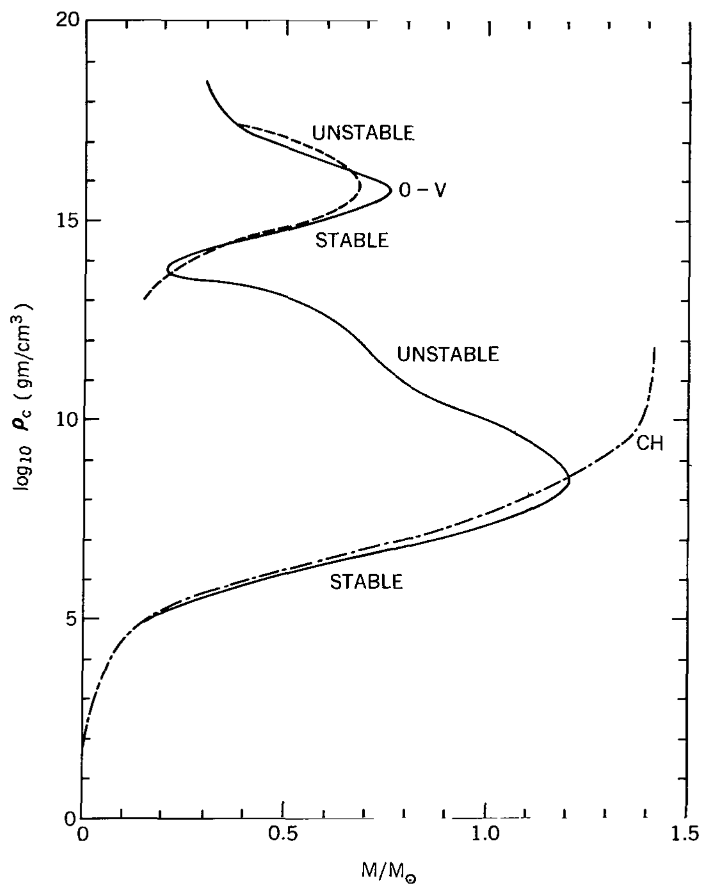

Figure 4—The mass-central density ρ_c relation for zero temperature stars. Given a central density, with the equation of state described in Figure 3 and Section 3.3, the mass may be integrated by using Equations 2.9 and 2.10 or Equations 5.2 and 5.3. Curves marked CH show the nongeneral-relativistic result of Chandrasekhar (Reference 17), and the curve marked O-V is the general relativistic result of Oppenheimer and Volkoff for an ideal Fermi gas (Reference 27). This solid curve is due to Wheeler et al. (References 19 and 25).

3.4. Neutrino Production at Near Zero Stellar Temperatures

We now turn to the energetics of dense matter at non-zero temperatures. As long as $kT << E_F$, the thermodynamic properties of matter are not different from that for the zero temperature case. Hence we shall only consider the case where $kT \approx E_F$. It may be pointed out that, as long as Γ is close to 4/3, the relation $T \propto \rho^{1/3}$ (Equation 2.28) is expected to hold for the temperature and density of a star during its evolution. If $E_F >> mc^2$, $E_F \propto \rho^{1/3}$. Hence the ratio kT/E_F will be roughly a constant if energy dissipation is not substantial; otherwise it will decrease.

In the following we shall assume that the ratio kT/E_F will be roughly 1. With this assumption, if we find the relaxation time for losing energy to neutrinos is short compared with the time for the star to restore to its original density, then we are back to the zero temperature case.

At $T = 6 \times 10^9$ °K ($kT \approx 0.5$ Mev) the most dominant neutrino process is the annihilation process (References 21, 22, and 23):

$$\gamma \rightleftarrows e^- + e^+ \longrightarrow \nu_e + \bar{\nu}_e \quad . \tag{3.18}$$

At higher temperatures then neutrino processes also must be considered. At low density $\left(\rho < 10^7\right.$ gm/cm^3, say$\left.\right)$ the mean free path of neutrinos is longer than the physical dimension of the star. At higher density, the mean free path of neutrinos is no longer compared with the physical dimension of the star. The rate of dissipation of neutrinal energy is then less than the production rate.

At $kT \approx m_\mu c^2$, the creation of π meson pairs and μ meson pairs is not negligible. Because the neutrinos associated with the μ meson do not interact with electrons to the lowest order, the dissipation of stellar energy through μ-neutrino production is more important. We shall also include this in our discussion.

(a) Electron-Positron Pair, π and μ Pair Densities

The number density for the electron pair created, under the condition $E_F/kT \approx 1$, is roughly given by (see, for example, Reference 23)

$$n_e = \frac{8\pi m_e^3 c^3}{h^3} \int_0^\infty \frac{x^2 dx}{\exp\left[(1 + x^2)^{1/2}/\theta_e\right] + 1} \,, \tag{3.19}$$

where

$$\theta_e = \frac{kT}{m_e c^2} = \frac{T}{T_e} \,. \tag{3.20}$$

Numerically

$$T_e = 6 \times 10^9 \,{}^\circ K \,. \tag{3.21}$$

Since $T >> T_e$, we may use the approximation $\left(1 + x^2\right)^{1/2} \approx x$. Changing variables from x to $u \equiv x/\theta_e$, we have

$$n_e = \frac{8\pi m_e^3 c^3}{h^3} \theta_e^3 \int_0^\infty \frac{u^2 du}{\exp u + 1} \,. \tag{3.22}$$

Since

$$\int_0^\infty \frac{u^2 du}{\exp u + 1} = \zeta(3) = 1.2202 \,, \tag{3.23}$$

where $\zeta(z)$ is the Riemann's zeta function, we find numerically that

$$n_e = 4.4 \times 10^{30}\, \theta_e^3 \ . \tag{3.24}$$

For the case of π or μ mesons we may define

$$\theta_\mu \equiv \frac{kT}{m_\mu c^2} = \frac{T}{T_\mu} , \tag{3.25}$$

$$\theta_\pi \equiv \frac{kT}{m_\pi c^2} = \frac{T}{T_\pi} , \tag{3.26}$$

$$T_\mu = 1.24 \times 10^{12}{}^\circ K \ , \tag{3.27}$$

$$T_\pi = 1.65 \times 10^{12}{}^\circ K \ . \tag{3.28}$$

If we neglect the factor unity in Equation 3.19, we have

$$n_\pi = 5.8 \times 10^{37}\, \theta_\pi K_2 \left(\frac{1}{\theta_\pi}\right) , \tag{3.29}$$

$$n_\mu = 1.54 \times 10^{37}\, \theta_\mu K_2 \left(\frac{1}{\theta_\mu}\right) , \tag{3.30}$$

where $K_2(z)$ is the modified Hankel function with the following asymptotic expression:

$$K_2(z) \longrightarrow \frac{2}{z^2} \qquad z \ll 1 , \tag{3.31}$$

$$K_2(z) \longrightarrow \sqrt{\frac{\pi}{2z}}\, e^{-z} \qquad z \gg 1 . \tag{3.32}$$

Table 2 lists the number density of electron pairs and μ meson pairs for $T = 10^{10}$°K to 10^{12}°K.

Table 2

Electron, π Meson, and μ Meson Number Densities as a Function of Temperature

(T is measured in °K; n in particles/cm³).

$\log_{10} T$	$\log_{10} n_e$	$\log_{10} n_\mu$	$\log_{10} n_\pi$
10	31.31	--	--
10.2	31.91	--	--
10.4	32.51	--	--
10.6	33.11	21.7	19.27
10.8	33.71	27.1	26.37
11	34.31	30.5	28.47
11.2	34.91	32.8	31.69
11.4	35.51	34.4	33.69
11.6	36.11	35.6	35.09
11.8	36.71	36.5	36.18
12	37.31	37.2	37.0
12.2	37.91	37.9	--
12.4	38.51	38.5	--
12.6	39.11	39.1	--

(b) Neutrino Cross Sections at High Energy

When the energy difference between neutron and proton can be neglected, the cross section for first order neutrino process is given by

$$\sigma \frac{v}{c} = C G^2 p_\nu^2 , \tag{3.33}$$

where G is the weak interaction constant, v is the relative velocity of the initial system of particles ($\approx c$), and p_ν is the final neutrino momentum. C is a numerical constant listed for a number of reactions in Table 3.

Table 3

Neutrino Reactions

	Reaction	$c \equiv \sigma/G^2P_\nu^2$
e - 1	$e^- + e^+ \rightarrow \nu_e + \bar{\nu}_e$	$1/3\pi$
e - 2	$e^- + p \rightarrow n + \nu_e$	$5.5/\pi$
e - 3	$e^+ + n \rightarrow p + \bar{\nu}_e$	$5.5/\pi$
e - 4	$e^- + \gamma \rightarrow e^- + \nu_e + \bar{\nu}_e$	α/π
μ - 1	$\mu^- + \mu^+ \rightarrow \nu_\mu + \bar{\nu}_\mu$	$1/3\pi$
μ - 2	$\mu^- + p \rightarrow n + \nu_\mu$	$5.5/\pi$
μ - 3	$\mu^+ + n \rightarrow p + \bar{\nu}_\mu$	$5.5/\pi$
μ - 4	$\pi \rightarrow \mu + \nu_\mu$	Lifetime $= 2.2 \times 10^{-8}$ sec
μ - 5	$\mu \rightarrow e + \nu_\mu + \nu_e$	Lifetime $= 2.2 \times 10^{-6}$ sec

3.5. Relaxation Time for Cooling of a Collapsing Star

At high temperature the energy difference between proton and neutron may be neglected. If the relation between the temperature T and the density ρ is given by*

$$\frac{T}{10^{12}\,{}^\circ K} = \left(\frac{\rho}{10^{13}\ \mathrm{gm/cm^3}}\right)^{1/3}, \tag{3.34}$$

the neutrino yield from electron-neutrino processes will be roughly the same, except the photo-neutrino process e - 4 which is down by a factor of $\alpha/\pi \approx 1/400$. The μ-neutrino processes show a wider variation. All μ capture or annihilation processes have roughly the same cross section as the electron process, as is expected from the theory. However, π and μ mesons can decay into electrons and neutrinos. The rate of decay, to the extent that degeneracy is neglected, is independent of the energy (apart from a time dilatation factor). Up to a temperature of $\approx 10^{12}\,{}^\circ K$, μ-neutrino emission from π-decay is the most important process.

As the density of the star increases, the mean free path of neutrinos becomes smaller. Once the mean free path of the neutrinos becomes smaller than the dimension of the star, the neutrino energy loss rate is no longer the same as the production rate. Let the radius of the star be R, the

*Equation 3.34 is consistent with the assumption that $E_F/kT \approx 1$.

mean free path for neutrinos be λ, the total stellar neutrino energy content be U_ν; then the neutrino luminosity L_ν for a star is

$$L_\nu = \frac{U_\nu}{\frac{R}{c} \cdot \frac{R}{\lambda}} . \tag{3.35}$$

Equation 3.35 has the following interpretation: When $\lambda = R$, the neutrinos leave the star with the velocity of light and the time constant for doing this is just R/c, the time of transit across the star with a velocity of light. When $\lambda < R$, neutrinos on the average scatter R/λ times before they leak out of the star. Its path length is approximately increased by a factor of R/λ. Hence Equation 3.35 describes the rate of decay of stellar neutrinal energy. The neutrinal energy content of the star may be estimated in the following way: The average energy of each neutrino is around kT. If all neutrinal states are filled up, further neutrino emission processes are inhibited. The total energy content allowable to neutrinos is of the same order as the energy content of the electron pairs, which is of the same order of magnitude as the total energy content.

A recent experiment demonstrated that the neutrino associated with electrons (e-neutrino) and the neutrino associated with μ mesons (μ-neutrino) have different quantum numbers (Reference 24). That is, a μ-neutrino cannot induce an inverse beta reaction:

$$\left.\begin{aligned} \bar{\nu}_\mu + p &\not\rightarrow n + e^+ , \\ \nu_\mu + n &\not\rightarrow p + e^- , \end{aligned}\right\} \tag{3.36}$$

nor can an e-neutrino induce inverse μ capture reactions:

$$\left.\begin{aligned} \bar{\nu}_e + p &\not\rightarrow n + \mu^+ , \\ \nu_e + n &\not\rightarrow p + \mu^- . \end{aligned}\right\} \tag{3.37}$$

Since the average energy of μ-neutrinos coming from π-decay and μ-decay is of the order of $m_\mu c^2$ (or less), inverse μ capture reaction cannot occur. Until $kT \approx m_\mu c^2$ ($T \approx 10^{12}\,°K$), the number density for μ mesons is small. At the temperature ($T = 1.6 \times 10^{11}\,°K$) when the loss of energy due to π decay becomes very important, the mean free path for ν_μ-μ scattering is 10^8 cm, which is around the radius of the star ($\approx 7 \times 10^7$ cm). Hence μ-neutrinos can dissipate stellar energy rapidly while e-neutrinos cannot. Table 4 summarizes our result. The relaxation time for dissipating stellar energy is shortest at roughly $T = 6.3 \times 10^{10}\,°K$, and again at $T \approx 4 \times 10^{11}\,°K$. The corresponding relaxation times are roughly 3×10^{-3} and 10^{-4} seconds, respectively. This is roughly the time scale for the star to relax to the zero temperature configuration discussed previously.

Since this time scale is of the same order of magnitude for light to travel across the star, we conclude the core collapses under zero temperature conditions ($E_F/kT >> 1$) . Colgate's proposal

Table 4

Neutrino Loss Rates in a Neutron Star.

Temperature (°K)	Density (gm/cm^3)	Energy Density ($ergs/cm^3$)	Neutrino Production Rate ($ergs/sec-cm^3$) ν_e	Neutrino Production Rate ($ergs/sec-cm^3$) ν_μ	Mean Free Path (cm) ν_e	Mean Free Path (cm) ν_μ	Radius of Star (cm)	Relaxation Time (sec) ν_e	Relaxation Time (sec) ν_μ
10^{10}	10^{7}	1.6×10^{26}	10^{26}	--	10^{10}	--	10^{9}	1	--
	4×10^{7}	10^{27}	3×10^{27}	--	1.6×10^{9}	--	6.3×10^{8}	3×10^{-1}	--
	1.6×10^{8}	6.3×10^{27}	2×10^{29}	4×10^{12}	2.5×10^{8}	> Star	4×10^{8}	3×10^{-2}	--
	6.3×10^{8}	4×10^{28}	1.3×10^{31}	7×10^{21}	4×10^{7}	> Star	2.5×10^{8}	3×10^{-3}	--
	2.5×10^{9}	2.5×10^{29}	10^{33}	9×10^{28}	6.3×10^{6}	> Star	1.6×10^{8}	0.1	3
10^{11}	10^{10}	1.6×10^{30}	Equilibrium	10^{31}	10^{6}	> Star	10^{8}	0.3	0.1
	4×10^{10}	10^{31}	Equilibrium	2×10^{34}	1.6×10^{5}	> Star	6.3×10^{7}	0.5	5×10^{-4}
	1.6×10^{11}	6.3×10^{31}	Equilibrium	2×10^{36}	2.5×10^{4}	> Star	4×10^{7}	1	3×10^{-5}
	6.3×10^{11}	4×10^{32}	Equilibrium	5×10^{37}	4×10^{3}	> Star	2.5×10^{7}	--	10^{-5}
	2.5×10^{12}	2.5×10^{33}	Equilibrium	Equilibrium	6×10^{2}	≈ Star	1.6×10^{7}	--	10^{-4}
10^{12}	10^{13}	1.6×10^{34}	Equilibrium	Equilibrium	--	≈ Star	10^{7}	--	10^{-3}

(Reference 4) that a shock wave will first compress the star into neutron star density, that it will be rebounded to restore the core to normal density, is somewhat unrealistic.

We have not included in our calculations the red shift of neutrinos due to the gravitational field of the star. The red shift caused by the gravitational field of the star will at most decrease the net loss energy rate by a factor of 2 (Section 8.2, and also Reference 48). If the red shift becomes much larger, then the star approaches a singular position which we shall soon discuss.

4. STATIC STRUCTURE OF WHITE DWARFS AT ZERO TEMPERATURE

Using Equations 3.6 and 3.7 as equations of state, Chandrasekhar (Reference 17) integrated Equations 2.9 and 2.10 to obtain the static structure of white dwarfs. His result is shown in Figure 4 together with Wheeler's, and Oppenheimer and Volkoff's, results. The most interesting feature is that the central density (and the central pressure) become infinite when the mass is still finite. The limiting mass he obtained is $5.75\,M_\odot/\mu_e^2$ where $\mu_e = \langle A \rangle / \langle Z \rangle$. Taking the most popular value of $\mu_e = 2$, the limiting mass is around $1.44\,M_\odot$.

Salpeter* worked out an intuitive derivation of the limiting mass, which is reproduced below. The virial theorem tells us that (Equation 2.20)

$$E_G \simeq E \ .$$

Let E_e be the kinetic energy per electron. Then

$$\left. \begin{aligned} E_G &\approx \frac{GM^2}{R} \ , \\ E &\approx NE_e \end{aligned} \right\} \tag{4.1}$$

and

$$M = N\mu_e m_p \ . \tag{4.2}$$

Hence

$$E_e = \frac{GM_p^2 \mu_e^2 N}{R} \ . \tag{4.3}$$

*Private communication.

Let r_e be the average value of the inter-electron spacing and p_e be the average momentum; then the uncertainty principle gives

$$r_e p_e \geqq \hbar \quad . \tag{4.4}$$

Considering only the equal sign case, we find, from the definition of E_e and Equation 4.4, that

$$E_e = \frac{p_e^2}{2m_e} = \frac{m_e c^2}{2}\left(\frac{r_o}{r_e}\right)^2 \qquad p_e \ll m_e c \quad ,$$

$$E_e = p_e c = m_e c^2 \left(\frac{r_o}{r_e}\right) \qquad p_e \gg m_e c \quad , \tag{4.5}$$

where $r_o = \hbar/m_e c$ is the Compton wavelength for an electron. Combining these two equations, letting $s \equiv r_e/r_o$, one can write

$$E_e = mc^2\left[\frac{1}{s + 2s^2}\right] . \tag{4.6}$$

Wheeler (Reference 25) has checked the accuracy of Equation 4.6 against the more complete formula Equations 3.6 and 3.7; he finds the maximum deviation is around 8 percent. Since $R \approx N^{1/3} r_e$, in substituting Equation 4.6 into Equation 4.3, one obtains the following expression:

$$\frac{1}{1 + 2s} = \left[\frac{Gm_p^2 \mu_e^2}{r_o mc^2}\right] \cdot N^{2/3} \equiv \mu_e^2 \left(\frac{N}{N_o}\right)^{2/3} , \tag{4.7}$$

where

$$N_o = \left[\frac{Gm_p^2}{\hbar c}\right]^{-3/2} \approx 2N_\odot \; ; \tag{4.8}$$

N_o is a characteristic number and $N_\odot$ is the number of particles of the sun. When $N < N_o$, it is possible to find a value of s such that Equation 4.7 is satisfied. When $\mu_e^3 N > N_o$, it is not possible to satisfy Equation 4.7 with a positive value of s. Hence, there exists a limiting value of mass beyond which no static equilibrium structure is possible.

Long before the density ρ becomes infinite, inverse beta reactions described in Section 3.3 will occur and Chandrasekhar's description will *no longer* be valid. However, this occurs at a mass very close to his limiting mass; this is shown in Figure 4.

We believe the structure of white dwarfs is well understood. Although the density and the pressure may become infinite at a finite mass, this divergence never occurs in nature as demonstrated by Wheeler (Figure 4). This is, however, not true in the case of neutron stars.

5. THE STATIC STRUCTURE OF NEUTRON STARS

5.1. The Equation of Stellar Structure

Since the numerical value for GM/Rc^2 (the ratio of gravitational energy to rest energy) for a neutron star is close to unity, it is necessary to consider the structure from a general relativistic point of view. Because of neutrino energy dissipation (Section 3.4), the temperature of the neutron core may always be taken to be zero. The pressure P may be taken to be a function of material density only (P and ρ are all measured in proper coordinates). Further, we consider only the static structure of neutron stars with complete spherical symmetry and no rotation. Under these circumstances, the stress energy tensor becomes

$$T_{\mu\nu} = \begin{pmatrix} -P & 0 & 0 & 0 \\ 0 & -P & 0 & 0 \\ 0 & 0 & -P & 0 \\ 0 & 0 & 0 & \rho c^2 \end{pmatrix} . \tag{5.1}$$

The equation of stellar structure is a solution of Einstein field equations with a time static and spherically symmetric metric (References 26 and 27). The equation is:

$$\frac{dP}{dr} = \frac{\left(\frac{P}{c^2} + \rho\right) G \left(4\pi r^3 \frac{P}{c^2} + m^*\right)}{r\left(r - \frac{2Gm^*}{c^2}\right)} \tag{5.2}$$

and

$$\frac{dm^*}{dr} = 4\pi r^2 \rho \; . \tag{5.3}$$

Neglecting $1/c^2$ terms, Equation 5.2 reduces to that for a non-relativistic gas sphere. At the radius of the star, m^* is actually the gravitational mass observed by a distant observer (by the gravitational field the star produces). In the following we denote the proper mass (the mass before it was assembled into a star) by m, the mass observed by distant observers by an asterisk*.

The analytical properties of Equations 5.2 and 5.3 have been discussed by Oppenheimer and Volkoff (Reference 27). They found that the boundary conditions for an ordinary star apply (e.g., at $r = 0, m^* = 0$, and the central pressure $P_c > 0$, etc.).

5.2. The Equation of State

In order that a unique solution for Equations 4.2 and 4.3 be possible for a given set of boundary conditions, P must be a known function of ρ . We shall discuss the equation of state $P(\rho)$ in greater detail. Because the interparticle spacing is of the order of 10^{-13} cm or less, which is roughly the size of elementary particles, it is not possible to exclude the structure of elementary particles from our discussion as in the case of the white dwarfs.

Before we proceed to the detailed structure of the equation of state, we would like to examine the limitations on the equation of state.

5.3. The Unimportance of Quantum-Gravitational Effects

Since the gravitational field inside a neutron star is extremely strong, one might wonder if quantum effects of the gravitational field may interfere with normal interactions among particles. However, quantum effects of gravitational field will, in general, not be important, as we shall show below.

A well-known theorem in Riemannian geometry* states that, in a given space with a non-Euclidean metric, it is always possible to find a coordinate transformation such that locally at any given point the metric $g_{\mu\nu}$ may be reduced to that for a Minkowskian space; that is,

$$g'_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix} . \tag{5.4}$$

Unless the $g_{\mu\nu}$'s change considerably over the dimension of elementary particles $(\approx \hbar/m_\pi c \approx 10^{-13}\text{ cm})$, it is not necessary to consider quantum effects of gravitational fields. Even in the extreme case when the mass of a neutron star is close to its critical mass, when the time

*This is easily proven. The line element ds may be written as:

$$ds^2 = \sum_{i,j} g_{ij}\, dx^i\, dx^j ,$$

where $\{g_{ij}\}$ is the metric tensor and $\{x^i\}$ is a general set of coordinates. Since $\{g_{ij}\}$ is a symmetric tensor, at a particular point $\{g_{ij}\}$ can be diagonalized by "rotating" the $\{x^i\}$ coordinate system at this point. An additional scale change in the rotated $\{x^1\}$ system reduces the line element to

$$ds^2 = \sum_{i,j} \pm \delta_{ij}\, dx^i\, dx^j ,$$

where δ_{ij} is the Kronecker delta and the sign (±) is chosen as to be consistent with the signature of Equation 13.1. Since g_{ij} varies with position, obviously this transformation is valid only at the particular point chosen.

metric g_{oo} vanishes at the center, quantum effects may still not enter. According to Oppenheimer and Volkoff one can write generally, in the center of a neutron star:

$$g_{oo} = \left(1 - \frac{2Gm^*}{Rc}\right)^{1/2} \exp\left[-\frac{1}{c^2}\int_o^{P_c} \frac{dP}{\frac{P}{c^2}+\rho}\right]; \tag{5.5}$$

when the mass of a neutron star approaches the critical mass, $P_c \to \infty$, the integral in the exponential diverges logarithmically as one approaches the center, and g_{oo} vanishes as $1/P$ $(P \to \infty)$. Unless $P \to \infty$ abruptly over a dimension of an elementary particle, g_{oo} cannot change substantially. This would not occur since the pressure is itself a macroscopic concept. Hence the center of neutron stars is not likely to be the dwelling place of quantum gravitational field theory.

It might be questioned that, since the trajectories of particles with spin are not geodesics (References 28 and 29), this might be an indication that quantum-gravitational effects play an important role. However, the spin-gravity interaction depends on the gradient of the gravitational field (spin-orbit type coupling) (References 28 and 29). Unless g_{oo} changes substantially over the dimension of an elementary particle, this interaction may be treated classically. Thus the criteria for replacing a gravitational field by an acceleration is the same for particles with spin or without spin. As long as the pressure retains its macroscopic meaning, locally we can always replace gravitational fields, however strong, by a uniform acceleration. In the following, we therefore neglect the quantum gravitational effects.

5.4. Relativistic Limits on the Equation of State

General relativity does set up certain limits for the equation of state on the basis of positive (or negative, depending on which convention to follow) definiteness of the stress energy tensor, and that signals cannot propagate at a speed faster than light speed.

The speed of sound v_s for a given medium is given as

$$v_s = c\sqrt{\frac{dP}{d\epsilon}} < c , \tag{5.6}$$

where ϵ is the energy density. Integrating Equation 5.6 one obtains

$$P < \epsilon . \tag{5.7}$$

That is, the pressure must not exceed the energy density.

This is a more relaxed limit for P.

In the case of a non-interacting ideal gas (including photon gas) the trace of the energy momentum tensor T_{ij} may be demonstrated to have a positive definite value (see, for example, Reference 30); T_{ij} is given by:

$$T_{ij} = \begin{pmatrix} -P & 0 & 0 & 0 \\ 0 & -P & 0 & 0 \\ 0 & 0 & -P & 0 \\ 0 & 0 & 0 & +\epsilon \end{pmatrix} . \tag{5.8}$$

Thus we have a more stringent condition on P:

$$P \leq \frac{1}{3} \epsilon \cdot \tag{5.9}$$

Only in the case of a photon gas one can have $P = \epsilon/3$. This sets an upper limit for the speed of sound to be $(1/\sqrt{3})c$.

Recently, however, Zel'Dovich (Reference 31) constructed an example of a relativistically invariant theory for a classical vector field with a mass, interacting with stationary classical point charges. In this theory the energy density ϵ and the pressure P are

$$\epsilon = nM + \frac{2\pi g^2 n^2}{\mu^2} , \tag{5.10}$$

$$P = \frac{2\pi g^2 n^2}{\mu^2} , \tag{5.11}$$

(units: $\hbar = c = 1$) where M is the mass of the heavy particle, μ the mass of the field quanta, g the charge of the field quanta, and n the number density of heavy particles. In the limit of large n, $\epsilon \propto n^2$ and $\epsilon \approx P$.

One might argue that, since for a particle with mass the stress energy tensor is

$$T_{\mu\nu} = \left(\rho + \frac{1}{c^2} P\right) U^{\mu} U^{\nu} - \frac{1}{c^2} \rho \eta^{\mu\nu} , \tag{5.12}$$

(where $\eta^{\mu\nu}$ is the Minkowskian metric, U^{μ} the four velocity) and that nothing can be "lighter" than a photon (for photon the rest mass is zero), $P = 1/3\, \epsilon$ should be the ultimate limit for all equations of state. Zel'Dovich's example indicates that such intuitive deduction is not correct.

5.5. Detailed Structure of Matter at High Density

The composition of matter at high density was considered in detail by Ambartsumyan and Saakyan (References 32 and 33). When the Fermi energy (including the rest energy) of neutrons exceeds the rest energy of strange particles, strange particles will be created in equilibrium with neutron gas. Although it is necessary to preserve the strangeness conservation law in the production process, in the decay process such a law need not be preserved and the number density for strange particles is governed by their ability to decay. Thus, we do not need to worry about the seemingly strangeness non-conserving character of reactions of the following type:

$$n + n \rightleftarrows \Lambda^\circ + \Lambda^\circ \ . \tag{5.13}$$

With this fact in mind, we now write down the reactions we will consider:*

$$\left.\begin{aligned} Y^\circ &\rightleftarrows n \ , \\ e^- + Y^+ &\rightleftarrows n \ , \\ Y^- &\rightleftarrows n + e^- \ , \\ e^- + P &\rightleftarrows n \ , \\ e^- &\rightleftarrows \mu^- \ , \\ e^- &\rightleftarrows \pi^- \ , \end{aligned}\right\} \tag{5.14}$$

where Y denotes one or more type of strange particles. Let the Fermi energy of particle x be denoted as E_x, its number density by N_x. Since at zero temperature the Fermi energy is the same as the chemical potential, the equilibrium condition for Equation 5.14 is:

$$\left.\begin{aligned} E_{Y^\circ} = E_n &= E_e + E_{Y+} = E_e + E_p \ , \\ E_{Y-} &= E_n + E_e \ , \\ E_{e-} = E_{\mu-} &= m_\pi c^2 \ . \end{aligned}\right\} \tag{5.15}$$

The reason for setting the chemical potential of π meson to its rest energy is because it is a boson which allows infinite condensation at zero kinetic energy. Further, the charge neutrality condition gives an additional constraint:

$$\Sigma N_{Y+} - \Sigma N_{Y-} - N_e - N_\mu - N_\pi = 0 \ . \tag{5.16}$$

No π^+, μ^+ are allowed because they will annihilate with π^-, μ^- to become photons (or neutrinos) which subsequently will escape.

*The present approach is due to Ambartsumyan et al. (References 32 and 33).

From Equations 5.15 and 5.16 the following conclusions may be drawn:

(1) The number density for e and μ are fixed independent of the material density once the Fermi energy of electrons reaches $m_\pi c^2$. At extremely high density only baryons contribute to the pressure. The threshold for π meson creation occurs at a baryon density of 5.86×10^{40} particles/cm^3, or a material density of 10^{17} gm/cm^3.

(2) π meson is the only boson that can be present. It represents an energy resevoir without contributing to the pressure. Other bosons created (e.g., the intermediate boson) will decay into π mesons directly or indirectly.

The composition of matter becomes very complicated at high energy. Particles become extremely densely packed. At a density of 10^{17} gm/cm^3 the interparticle spacing is around 4×10^{-14} cm, which is about the radius of the repulsive core for protons and neutrons. Hence at such high density the interaction among particles is by no means negligible. The interacting energy is around 10 Bev in the cm system (which corresponds to a laboratory energy of 100 Bev). Experimental study of interactions at such a high energy is somewhat beyond the present capability. Our knowledge of the equations of state for closely packed neutron matter is therefore highly uncertain.

5.6. Models for Neutron Stars

We have seen that nuclear interactions among elementary particles at extremely high energy play an important role in the macroscopic behavior of dense neutron stars. In the following we shall demonstrate that there may exist a paradox between a very fundamental conservation law (baryon conservation law) and general relativity. From neutron star models with a hypothetical equation we may study the properties of this paradox which, if resolved, should yield us important information on extremely high energy interactions.

We are mostly interested in the singular character of the metric g_{oo}. In the following we shall review work done by other physicists on this subject.

(a) Oppenheimer-Volkoff Theory

The very first relativistic model for neutron stars was constructed by Oppenheimer and Volkoff (Reference 27). They assumed an equation of state of an ideal Fermi gas. Their mass-radius relation is demonstrated in Figure 4. The equilibrium mass m^* (Equation 5.3) as a function of density shows a maximum of 0.76 $M_\odot$ at a density of around 10^{16} gm/cm^3 and then declines to 0.3 $M_\odot$ when the pressure P and the density ρ diverge at the center.* Since at the center $m^* = 0$, we have, from Equation 5.5,

$$g_{oo} = \exp\left[-\frac{1}{c^2}\int_o^{P_c} \frac{dP}{\frac{P}{c^2} + \rho}\right], \tag{5.17}$$

*The masses we talk about are masses observed by observers at large distances.

g_{oo} approaches zero when $P_c \to \infty$. Thus a singularity develops at the center when the mass is still finite. This singularity, unlike the Schwarzschild singularity, *cannot* be removed by a mere coordinate transformation (see Section 6). It may be a real singularity.

*(b) Cameron's Model**

Following Oppenheimer-Volkoff's model, several other models have been constructed. Cameron (Reference 10), in particular, used an equation of state computed by Skyrme according to many-body theory, taking into account the nuclear interaction at low energy. Salpeter worked out a more complete equation of state computed on the basis of effective range theory. It agrees with Skyrme's theory in the low energy limit for which Skyrme's theory applies. Cameron's result is shown in Figure 5. Again, a singularity ($g_{oo} = 0$) develops at the center of his model star at a finite mass. The value of the critical mass is around 2 $M_\odot$.

(c) Wheeler et al. Hard Sphere Model

Motivated by the use of repulsive core to interpret low energy nuclear interactions, Wheeler et al. (Reference 19) studied the extreme case: the incompressible fluid for which the equation of state is

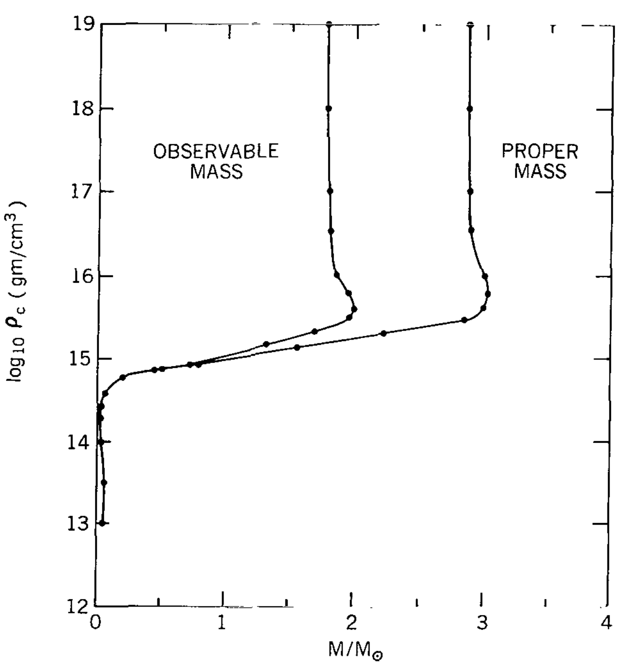

Figure 5—The mass-central density ρ_c relation in Cameron's model (Reference 10). Observable mass is m*; the proper mass is m as defined in Section 5.1.

$$\left.\begin{aligned} P &= \text{finite} \\ P &= \infty \end{aligned}\right\} \rho < \rho_o , \qquad (5.18)$$

where ρ_o is some density. Despite this kind of equation of state if the mass is not too large, in general no singularities will appear; but, for a given value of ρ_o, there exists a critical mass for which a singularity ($g_{oo} = 0$) appears at the center. Moreover, they found that

$$\frac{dM^*}{dM} = \left[1 - \left(\frac{M^*}{M_{cr}}\right)^{2/3}\right]^{1/2} , \qquad (5.19)$$

where M* is the mass of the star observed by distant observer, M is the mass of the material that composes the star before the star is assembled, and M_{cr} is a critical mass depending on the value of ρ_o. Hence $dM^*/dM = 0$

*G. S. Saakyan (Reference 34) recently has repeated the computation by Cameron as described here. He found a mistake in Cameron's paper, and that the value of the critical mass is not as high as Cameron claimed.

when $M^* = M_{cr}$. Moreover, they found that the ratio

$$\frac{M^*}{M}$$

has a non-zero finite value even when $M^* = M_{cr}$. We shall explore the meaning and implications of this result in the following discussions.

(d) Ambartsumyan-Saakyan Model

Ambartsumyan and Saakyan (References 32 and 33) studied the neutron star models on the basis of their equation of state, taking into account strange particles. Their result is shown in Figure 6.

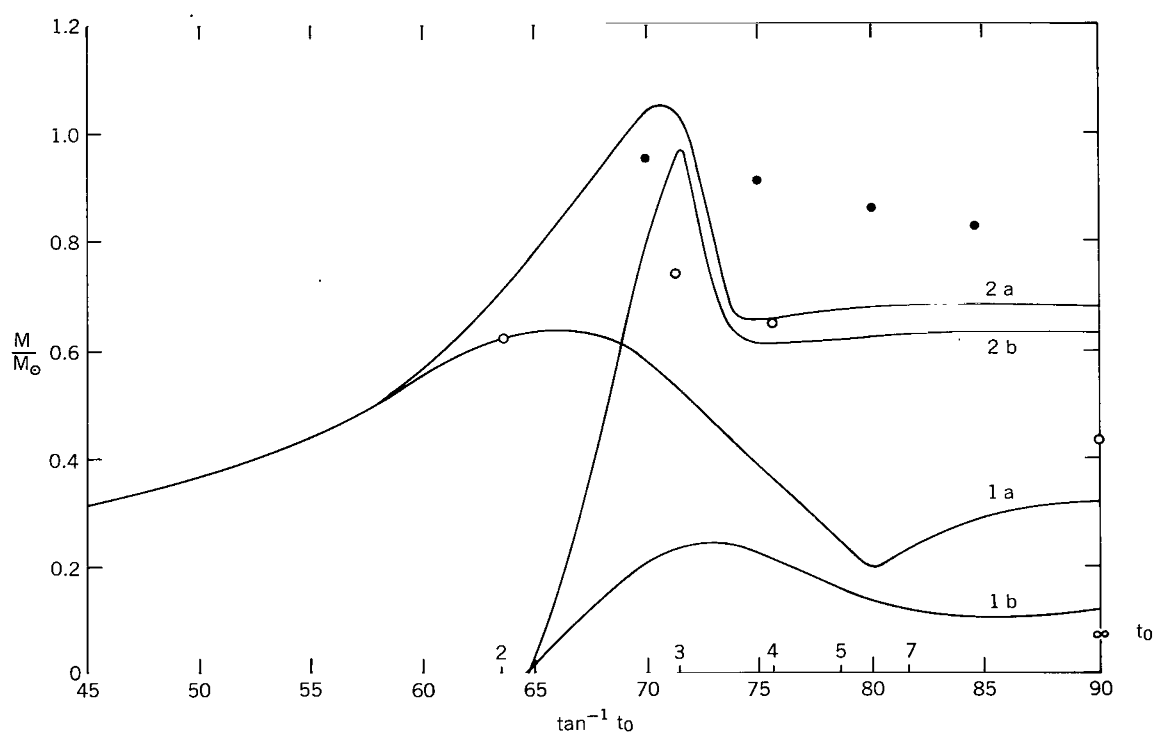

Figure 6—The mass-t relation in Ambartsumyan and Saakyan's model; t_o is the value of t at the center of the star, where

$$t = 4 \ln \left\{ \frac{p_F}{m_n c} + \left[1 + \left(\frac{p_F}{m_n c} \right)^2 \right]^{1/2} \right\} .$$

Curves Ia and Ib depict the mass of the star and its hyperon core in the case where elementary particles form an ideal gas at any densities. Curves 2a and 2b present the same picture for the case where repulsive forces acting between baryons at high densities (real gas) are taken into account. Black dots indicate the mass of configurations consisting of neutrons, with repulsive forces taken into account. Hollow dots indicate the mass of those configurations consisting of an ideal neutron gas. (Calculations by Oppenheimer and Volkoff, Reference 27.)

Essentially there is no change in the conclusion that g_{oo} vanishes at the center at a finite mass.

6. DYNAMIC COLLAPSE OF OVERSIZED NEUTRON STARS

Although the four treatments of neutron star models have not exhausted the possibility of all types of equation of states, they do indicate that inevitably g_{oo} vanishes at the center for a finite neutron star mass. The question arises: Is this singularity unavoidable? Is there no way to save the face of physics by avoiding infinite pressure and density? In this section we would like to examine the problem of the dynamical collapse of a star more closely, and the nature of the singularity with which we are faced.

6.1. Schwarzschild Singularity and the Ultimate Mass Limit

For a point mass, the Schwarzschild solution for the metric has the form (Reference 26):

$$ds^2 = g_{ij}\, dx^{ij} = -\frac{dr^2}{1 - \dfrac{2GM}{rc^2}} - r^2\, d\theta^2 - r^2 \sin^2\theta\, d\phi^2 + \left(1 - \frac{2GM}{rc^2}\right) dt^2 . \tag{6.1}$$

For a mass m, a singularity occurs at $r_s = 2GM/c^2$. That is, $ds^2 = \infty$. Consequently no light signal can cross this boundary defined by $r = r_s$.* For a proton, $r_s \approx 10^{-53}$ cm. Since the radius of a proton is 10^{-13} cm, it does seem that the Schwarzschild singularity may not play as important role in the structure of a proton. For massive objects like the sun, the Schwarzschild radius is around 2.6 km, which is about the size of a neutron star of the same mass with a density of around 10^{16} gm.

This singularity has been a subject of discussion among relativists ever since the solution was found. Is it a real singularity? Or is it just a singularity related to a particular coordinate system?

By a change of coordinates of $r \to \bar{r}$, $\theta \to \theta$, $\phi \to \phi$, $t \to t$, where $\bar{r}$ is related to r by the following relation†:

$$r = \left(1 + \frac{GM}{2\bar{r}c^2}\right)^2 \bar{r} . \tag{6.2}$$

Equation 6.1 becomes

$$ds^2 = -\left(1 + \frac{m'}{2\bar{r}}\right)^4 \left(d\bar{r}^2 + \bar{r}^2\, d\theta^2 + \bar{r}^2 \sin^2\theta\, d\phi^2\right) + \frac{(1 - m'/2\bar{r})^2}{(1 + m'/2\bar{r})^2}\, dt^2 , \tag{6.3}$$

*We might remark that such behavior was first predicted by Laplace on the basis of a corpuscular theory of light. See Reference 35.
†This coordinate system is known as the isotropic coordinate system.

where

$$m' = \frac{GM}{c^2}$$

is the geometrized mass. In Equation 6.3 the singularity in $\bar{r}$ coordinates disappears. Hence it seems that the singularity at $r = r_s$ is a coordinate singularity.

Further, Robertson (Reference 36) has demonstrated that, by a coordinate transformation, one can eliminate this singularity in the following sense: An observer in the flat space observes the free fall of a test particle towards the singularity. The time it takes the test particle to cross the boundary (which is at $r_s = 2GM/c^2$) is infinite; this is because $g_{oo} = 0$ at the boundary and the test particle suffers infinite time dilatation as it approaches the singularity. However, if the observer moves with the test particle, the time duration that he observes for the test particle (and himself) to cross the boundary is *finite*. Hence, in the opinion of many relativitists, this singularity is a *coordinate* singularity since it can be removed by a proper choice of coordinate systems (transformation to a co-moving frame).

We regard the Schwarzschild singularity as a *physical singularity* in the following sense. Given a matter distribution such that initially nowhere within the matter distribution is $r_s \leq 2GM/c^2$. An observer observes this distribution of matter in a nearly flat space (since almost all points of the universe are locally nearly flat). Since it takes an infinite time for this distribution of matter to cross the Schwarzschild singularity, we may regard the Schwarzschild singularity to be a real, physical singularity. Once matter is found in a non-singular state, within the finite lifetime of the universe it will remain in the non-singular state. Similarly, if a Schwarzschild singularity were found in nature, it was created with the universe.

Since all massive stars undergo supernova explosion, their cores will inevitably collapse to extremely high densities. It might be worthwhile to investigate whether they collapse asymptotically intowithin Schwarzschild singularity or not. It is of more importance to investigate, first, whether a singularity similar to Schwarzschild singularity exists in neutron stars with all possibilities of equation of state.

As an interesting episode, we quote a paragraph from Eddington's momumental work on "The Internal Constitution of the Stars" (Reference 37):

> "The great bulk of these giant stars is due to low density rather than great mass.... It is rather interesting to notice that Einstein's theory of gravitation has something to say on this point. According to it a star of 250 million km radius could not possibly have so high a density as the sun. Firstly, the force of gravitation would be so great that light would be unable to escape from it, the rays falling back to the star like a stone to the earth. Secondly, the red-shift of the spectral lines would be so great that the spectrum would be shifted out of existence. Thirdly, the mass would produce so much curvature of the space-time metric that space would close up round the star, leaving us outside (i.e., nowhere). The second point gives a more delicate indication and shows that the density is less than 0.001; for even at that density there would be a red-shift of the spectrum too great to be concealed by any probable Doppler effect.
>
> Lest this argument should be regarded by our more conservative readers as ultra-modern, we hasten to add that it is to be found in the writings of Laplace*:

*Reference 35.

'A luminous star, of the same density as the earth, and whose diameter should be two hundred and fifty times larger than that of the sun, would not, in consequence of its attraction, allow any of its rays to arrive at us; it is therefore possible that the largest luminous bodies in the universe may, through this cause, be invisible.' "

6.2. Will a Massive Star Collapse into a Neutron Star Without Being Singular?

We have surveyed the work of many authors (which by no means exhausted all possible equations of state) and we find that it seems inescapable that g_{oo} will vanish at the center of a neutron star if its mass approaches a certain limit, which is around 1 $M_{\odot}$.

There are several properties characteristic to this kind of treatment of a neutron star:

(1) The rest energy of a particle is $\sqrt{g_{oo}}\,mc^2$ and at $g_{oo} = 0$ it takes no energy to create a pair of particles, nor does one gain any energy by annihilating a pair of particles. All photons emitted at points where $g_{oo} = 0$ will suffer infinite red shift and there will be no energy associated with such photons.

(2) The ratio of M^*/M_f is a well-defined number even when $M^* = M_{cr}$ in all cases. In Wheeler's work (Reference 19) this ratio is 2.8, and in Cameron's work (Reference 10) this ratio is 1.5.

(3) Although only in the simple incompressible fluid case can one calculate $dM^*/dM = 0$ at $M^* = M_{cr}$, we believe the result $dM^*/dM = 0$ at $M^* = M_{cr}$ is universal because, in the case of incompressible fluid, this is due to $g_{oo} = 0$ at the center. Thus consider a star at the critical mass. If one adds to it one more gram of mass, this mass will not add to the observed mass of the star, but will be converted into energy — into a kind that can escape, namely, radiation.

6.3. Wheeler's Theory

One may be perturbed by the fact that the star now becomes a gravitational machine by which matter is converted into energy, thus violating the baryon conservation law. Gravitational machines converting mass into energy are nothing new. Any gravitating body, in some sense, is a gravitational machine in the above sense. If we drop matter from infinity to the surface of the sun; roughly 10^{-6} of its mass will be converted into energy which escapes, and this piece of matter will be gravitationally bound to the sun. However, if we pump energy into the sun to disintegrate it, we will reobtain nucleons. There is no ambiguity.

In the case of neutron stars this does not seem to be so. At the critical mass limit the ratio of M^*/M is a fixed constant of the order of 1/3. According to Equation 5.19, the stellar mass approaches the critical mass when the original mass is finite. Beyond this point the star becomes a gravitational machine converting mass into energy. Thus the total number of nucleons is not a well-defined number. When we pump energy into the system to dissociate the star from its gravitational binding, we do not get all nucleons back — we only get a definite number $\left(\approx 3 \cdot M_{cr}/m_p\right)$ back. Thus, the baryon number may not be a meaningful concept for a neutron star.

From this example Wheeler (References 19 and 25) concluded that the baryon number conservation law must break down in a very dense star. We are open-minded to this kind of suggestion. Although the baryon conservation law has been demonstrated to high precision to be a valid concept in an ordinary nuclei (Reference 38), it has always been postulated only as an empirical law (e.g., Reference 39).

The origin of this law is still unclear. Moreover, we do not know what this law actually means when baryons are closely packed to within their structure ($\lesssim$ repulsive core radius). It certainly poses an extremely interesting and meanwhile difficult problem. The final solution, as we have seen in Section 5.3, must come from extremely high energy physics.

6.4. Oppenheimer-Snyder Theory

Oppenheimer and Snyder (Reference 40) had approached the problem of collapse in the following manner: In the absence of the pressure, the collapse may be treated analytically. They computed the world trajectories of material bodies falling towards a common center under the gravitational field produced by these material bodies. They obtained the following result: To an external observer these particles fall toward the Schwarzschild singularity asymptotically and the total time of fall is infinite. This is because of the infinite time dilatation effect mentioned earlier. To a local observer, the time is finite; and, for the case of the sun, it is of the order of 1 day. Since the star continues to fall indefinitely, the star is always in some dynamical motion and this is consistent with the fact that no static structure for such a star exists if its mass exceeds a certain limit.

We do not regard this resolution as satisfactory, since the pressure P cannot be neglected in the case of a neutron star; from what others have demonstrated (References 19, 31, 32, and 33), $P \approx \rho c^2$ and therefore cannot be neglected. Their solution, however, may be of cosmological significance. We shall discuss further possible solutions to this problem in Section 9.

7. ANGULAR MOMENTUM AND STELLAR COLLAPSE

We have delayed the discussion of the effect of angular momentum up to now — not because it is not important, but because it leads to very interesting consequences: namely, the possibility of gravitational radiation from a rapidly rotating neutron star, and the possibility of their detection.

The moment of inertia of a spherical body of radius R, mass M, and of uniform density ρ is:

$$I = \frac{2}{5} MR^2 . \tag{7.1}$$

The angular momentum $I\omega$ is a conserved quantity (where ω is the angular velocity). For a normal star $R \approx 10^{11}$ cm and the period of rotation is around 1 day. Thus,

$$I\omega \propto R^2 \omega \approx 6 \times 10^{17} . \tag{7.2}$$

The centrifugal force f_c per unit mass at the equator of the star is

$$f_c = \frac{v_{eq}^{2}}{R} ,$$

where $v_{eq} = R\omega$ is the speed at the equator. Since $f_c < MG/R^2$ (otherwise the star will be torn apart by rotation), we have the upper limit on R:

$$\frac{\left(R^2 \omega\right)^2}{R^3} < \frac{MG}{R^3} \tag{7.3}$$

or

$$R > \frac{\left(R^2 \omega\right)^2}{MG} . \tag{7.4}$$

This is the minimum radius to which a rotating star may collapse. Using $M = 20\ M_{\odot}$, and the value of $R^2 \omega$ given by Equation 7.2, we find that

$$R > 10^8 \text{ cm} , \tag{7.5}$$

and at this limiting value of R the angular velocity ω is around 1 rad/sec, and the period is around 0.1 sec.

Although a rotating body with cylindrical symmetry does not radiate gravitational waves, there is no reason that a rapidly rotating body should preserve its cylindrical symmetry. The classical Jacobi ellipsoid is an example: a homogeneous, uniform density fluid held together by its gravitational field can assume the shape of an ellipsoid with the axis of rotation perpendicular to the long axis of the ellipsoid. As an estimate we apply to our rotating star the formula of gravitational wave radiating power computed for a spinning rod (References 41, 42, and 43). The radiating power is

$$P = \frac{32GI^2 \omega^6}{5c^5} . \tag{7.6}$$

The rotational energy is $1/2\ (I\omega^2)$. Hence the relaxation time τ for the star to lose its angular momentum is

$$\tau = \frac{5c^5}{64GI\omega^4} . \tag{7.7}$$

For the parameters we consider, we find $\tau \approx 50$ seconds. Although this is longer than the time of dynamical collapse as calculated by Colgate (Reference 7), it is quite small compared with the lifetime of a neutron star, which we shall compute.

Weber thinks such gravitational radiation is detectable (References 41 and 44).

8. OBSERVABLE FEATURES OF NEUTRON STARS

8.1. Energy Content of a Neutron Star

We now shall calculate the energy content of neutron stars. The Fermi energy of neutrons (and hyperons) averaged over the star cannot be greater than $m_n c^2$; otherwise the gravitational energy of the star will exceed its rest energy and the star will be within its Schwarzschild radius. Further, if the temperature is too high, neutrino processes will quickly dissipate its thermal energy. Hence the temperature must be below a certain limit. From these criteria the temperature of neutron stars is thus estimated to be 10^9°K, assuming they were formed earlier at a high temperature. The relaxation time for cooling by neutrino emission at this temperature by plasma neutrino process and pair annihiliation process (References 21-23 and 45) is greater than 10^3 years, which will be the lifetime of neutron stars against optical emission (see next section).

We now apply the theory of a nearly zero temperature ideal Fermi gas to find the thermal energy. The specific heat per unit mass is given as Reference 17 .

$$c_v = \left(\frac{\partial \epsilon}{\partial T}\right)_v = \frac{\pi^3 k}{m_n}\left(\frac{kT}{m_n c^2}\right)\frac{x\left(x^2+1\right)^{1/2}}{x^3} \equiv \frac{T}{T_o}\,\frac{x\left(x^2+1\right)^{1/2}}{x^3}\,, \tag{8.1}$$

where $x = p_F^{(n)}/m_n c^2$ and $p_F^{(n)}$ is the Fermi momentum for the neutron gas. The total thermal energy ϵ_{th} is given by

$$\epsilon_{th} = \int_o^T c^{\sigma}\, dT = \frac{1}{2}\frac{T^2}{T_0}\,\frac{x\left(x^2+1\right)^{1/2}}{x^3}\,. \tag{8.2}$$

As we have remarked before, the overall average of E_F is less than $m_n c^2$. Therefore x is never very much different from unity. Taking $T = 10^9$°K and $x = 1$, we find that

$$T_o^{-1} = 0.75 \times 10^{-4} \tag{8.3}$$

and

$$\epsilon_{th} \cong 4.6 \times 10^{13}\ \text{ergs/gm}\,. \tag{8.4}$$

The total thermal energy of the neutron star is $\approx M \cdot 10^{14} = 10^{47}$ ergs, which is about the thermal energy of the sun.

The neutron core is not composed of neutrons alone. Hence the thermal energy is a sum over all particle energies. In this case, the value of x will be somewhat smaller than 1. The thermal energy of a neutron star may be several times greater than the value estimated here.

8.2. Physical Properties of the Surface of a Neutron Star

The surface of a neutron star must consist of ordinary matter since, when $\rho < 10^{12}$ gm/cm^3, the composition of matter is predominantly iron group elements. Thus we can picture a neutron star as a centrally condensed neutron core, surrounded by a layer of nearly iron group elements.

If the interior temperature of a neutron star is taken to be 10^9 °K, then when $\rho > 10^6$ gm/cm^3 the iron group elements will become degenerate. Since the thermal conductivity of a neutron core and the degenerate layer is very high, we need only consider the nondegenerate ordinary-stellar-matter envelope which contributes mostly to the overall opacity of the star.

The iron group elements have very high values of Z and the ionization temperature in the k-shell electrons is around 0.015 Mev, which corresponds to a temperature of 10^8 °K. The absorption coefficient exhibits a very sharp edge (absorption edge). At the ionization energy the absorption cross section may be as high as 10^{-20} cm^2. On the average, the mean free path of radiation may be as short as 0.1 gm/cm^2. When $T > 10^8$ °K, the main source of opacity is due to Compton scattering. The opacity is a constant. The mean free path of photons is roughly 5 gm/cm^2.

As a rough estimate we now apply the theory of white dwarf atmosphere to a neutron star (Reference 46). Assuming the main source of opacity to be due to the absorption edge of k-shell electrons, the radial dependence of temperature and density is:

$$\rho \propto \left(\frac{R}{P} - 1\right)^{3.25} , \tag{8.5}$$

$$T \propto \left(\frac{R}{P} - 1\right) , \tag{8.6}$$

where R is the radius of the neutron star and the proportional constant is determined by the mean density of the star. For a neutron star of mean density $\langle\rho\rangle = 10^{15}$ gm/cm^3, R is around 10^6 cm. Thus

$$\rho \approx \langle\rho\rangle \left(\frac{R}{r} - 1\right)^{3.25} = 10^{15} \left(\frac{\Delta r}{R}\right)^{3.25} . \tag{8.7}$$

Taking $\rho = 10^6$ gm, we find $\Delta r = 10^{-2.77}\, R \approx 10^3$ cm. Material contained in the envelope in gm/cm^2 is *roughly* $\int_0^{\Delta r} \rho\, dr \approx 10^8$ gm/cm^2. The number of scatterings a photon suffers in leaving the envelope is roughly $\int_0^{\Delta r} \rho\, dr\ \lambda \approx 10^8$, where λ is taken to be 5 cm/gm-cm^2.

To estimate the luminosity, we use an intuitive theory due to J. Steinberger.* Consider the following experimental setup: A series of parallel infinite plates of perfect blackbody properties is set up as in Figure 7. Let the temperature of the nth parallel plate be T_n, and let T_1 be fixed. The energy flow is assumed to be completely radiative. When a steady state of energy flow is reached, it is easy to verify the following relation:

$$\begin{aligned} acT_1^4 - acT_2^4 &= acT_2^4 - acT_3^4 = \cdots \\ &= acT_{n-1}^4 - acT_n^4 = \cdots \\ &= acT_{m-1}^4 - acT_m^4 = acT_m^4 . \end{aligned} \tag{8.8}$$

*Private communication.

where a is the Stefan-Boltzmann constant and c the velocity of light. From this equation we obtain a relation between T_1 and T_m:

$$acT_1^4 = \frac{acT_m^4}{m} \quad (8.9)$$

Photons are continuously absorbed and reemitted by atoms or nuclei. In the atmosphere of a star the atoms are like the parallel plate in Steinberger's model for radiative transfer. The energy flux radiated from a unit area on the surface of a neutron star is ($T_1 \approx 10^9$°K)

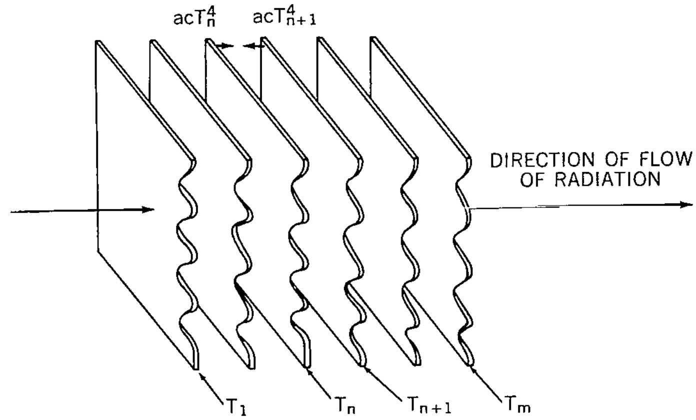

Figure 7—An extremely simplified model of stellar atmospheres. The plates are made from materials with perfect blackbody properties.

$$f = \frac{acT_1^4}{14 \times 10^8} \approx 5 \times 10^{23}\ \mathrm{ergs/cm^2\text{-}sec} \ . \quad (8.10)$$

The overall luminosity L is thus

$$L = 4\pi r^2 f \cong 10^{37}\ \mathrm{ergs/sec} \ . \quad (8.11)$$

The maximum of the spectral frequency corresponds to $T = 10^7$ °K. We have computed the energy content of a neutron star to be 10^{47} ergs. A neutron star radiating energy at such a lavish rate will last for at most 1000 years. However, as the temperature is reduced, the rate of energy radiation will be reduced, too; when $T_1 = 10^8$ °K, the luminosity L will be reduced to around 10^{-4} of the value given in Equation 8.11 and the lifetime will be around 10^4 years, with a maximum on the spectral frequency $T \approx 10^6$°K (0.1 kev).

8.3. Observational Problem

Our atmosphere is transparent only to radiation in the following wavelength region (e.g., Reference 47): 3000A→7000A (visible), 8000A→12,000A (infrared window), 1 cm→10 meters (microwave), and 200 meters→infinity (low frequency radio wave).

Interstellar matter consists mainly of hydrogen, and it is extremely opaque to radiation of wavelength near the Lyman continuum (≈1000A). Since the absorption coefficient exhibits a sharp edge and has a λ^3 dependence, for frequencies other than that near the Lyman continuum it is relatively transparent.

The surface temperature of neutron stars is around $10^6 \longrightarrow 10^7$ °K. Burbidge (Reference 48) has estimated the red shift and he found that, if the neutron stars have a very small envelope (as we have calculated), the light from such stars may suffer a red shift as large as $\lambda/2$. Hence the temperature observed by distant observers will be one-half the value at the surface. The red shift will therefore reduce the observable temperature by at most a factor of 2.

The maximum in Planck's formula for $T = 10^6 \longrightarrow 10^7$ °K is between 100A and 10A (in-between UV and x-ray bands). Both lie in a region where interstellar absorption is not large. The mean free path is around 10^5 light years, assuming an interstellar hydrogen density of 1 particle/cm^3. We remark that this wavelength region is very difficult to observe experimentally.

The fraction of this radiation energy R_v that will pass through our atmospheric window is easily seen to be given by

$$R_v = \frac{\int_0^x \frac{x^3\, dx}{e^x - 1}}{\int_0^\infty \frac{x^3\, dx}{e^x - 1}} = \frac{15}{\pi^4} \int_0^x \frac{x^3\, dx}{e^x - 1} , \tag{8.12}$$

where $x = h\nu/kT$. For small values of x, since $e^x \approx 1 + x$, we have

$$R_v = \frac{5}{\pi^4} x^3 . \tag{8.13}$$

For $kT = 50$ ev ($T \approx 5 \times 10^5$ °K), $h\nu = 5$ ev (corresponding to $\lambda \approx 3000$A), we find that

$$R_v \approx 5 \times 10^{-5} . \tag{8.14}$$

Thus, for a neutron star of internal temperature 10^8 °K it will have a luminosity of $\approx 10^{-6} L_\odot$. For such low luminosity stars, the star has to be, borrowing a comment from Greenstein,* "practically inside the solar system in order to be observable by the world's largest telescope, the 200 inch Mount Palomer telescope." Hence, it is not surprising that no neutron stars have ever been found.

Since in our picture every supernova star will inevitably become a neutron star, we can estimate their numbers in our galaxy. Taking the age of our galaxy to be 10^{10} years, and the rate of a supernova explosion to be roughly 10^{-2}/year (References 8 and 9), we find the total number of neutron stars in our galaxy to be 10^8, of which possibly ten will have a surface temperature of about 10^7 °K, and possibly 10^2 will have a surface temperature of about 10^6 °K.

*Private communication. The author is indebted to him for an enlightening discussion on the nature of peculiar radio sources which were once considered to be neutron stars. However, J. Greenstein demonstrated that they cannot be neutron stars (Reference 49).

With the use of proper instruments mounted in an earth satellite now available for scientific research, such neutron stars should be readily detectable. (See footnote on page 41, however.)

Moreover, since most stars possess rotation, the rotational energy will be dissipated, during the collapse phase, into gravitational waves. Such waves should be detectable. Instruments designed by Weber and his associates (References 41 and 44), if perfected to their expectation, should be able to detect such waves many galaxies away. Thus their instruments may be used to detect supernova explosions. We remark that their instruments are more sensitive than the neutrino monitor station suggested previously (References 1 and 21).

9. DISCUSSION

The central point of the paradox presented by Wheeler is essentially the following: Given a neutron star, assuming one can add to its mass (hence the nucleon numbers) until it reaches the critical value; then at this critical value a singularity $(g_{oo} = 0)$ develops at the center. Unlike the Schwarzschild singularity this singularity cannot be removed by performing a coordinate transformation. Further addition of mass to the star will not increase its mass nor its nucleon number. All nucleons added to it will be converted into radiation energy. This essentially violates the empirical law that baryons cannot be destroyed in any reaction. Since the existence of the critical mass is inevitable, Wheeler concluded that the baryon conservation law must be violated in superdense stars.

We have examined this problem and our conclusions are the following:

(a) The absence of quantum-gravitational effects: Inside a neutron star the gravitational field has to interact with the elementary particle field classically. This is because the gradient of the gravitational field is extremely small as compared with the gradients of nucleon, electromagnetic, etc. fields. By the principle of equivalence, one can replace a uniform gravitational field by a uniform acceleration. Thus the equation of state may be computed without regard to any gravitational field present.

(b) Wheeler argued that one cannot get away from this difficulty by proclaiming that the whole star falls through the Schwarzschild singularity. The problem poses, as he calls it, "a problem of principle." No one can forbid an observer from adding one extra gram of matter to a star already at the critical mass.

We would like to remark that it is by no means clear how this singularity is developed. It depends strongly on the equation of state. On the central density-mass curve (Figure 4) a maximum is found. Beyond this maximum, in most cases the gravitational binding energy of the neutron star is *positive*. This means *the critical mass cannot be reached statically*. Moreover, we know that, if the mass decreases with increasing central density, the stellar configuration is not a stable one. Hence, then the problem of the critical mass must be treated dynamically. We also note that the maximum in the mass is reached when the star is still free from singularity.

If the existence of the mass maximum is inevitable, the static problem as posed by Wheeler is irrelevant to the dynamical collapse of a star (the only way a neutron star may be formed).

Neglecting the pressure term in the stress energy tensor, Oppenheimer and Snyder (Reference 40) found that the star collapses toward the Schwarzschild singularity asymptotically. In reality the pressure cannot be neglected — in fact, $P \approx \rho c^2$. The essential questions are therefore: If one takes into account the pressure, does a real singularity such as $g_{oo} = 0$ develop at the center first? Or does the star still fall through the Schwarzschild singularity asymptotically? These are certainly relevant questions to ask. They can be answered theoretically by a further study of neutron star models.

(c) Another type of equation of state: S. Weinberg* posed the following argument. An incompressible fluid may be regarded as composed of particles with a repulsive potential between them. Although they are the hardest equations of state one can think of, they also carry an indefinitely large amount of energy. Since the *source* of gravitational field is energy itself, one gets into infinite trouble by letting the energy be infinitely large. He therefore suggested the following: Let there be an attractive force of small range. When matter is compressed hard enough, the attractive force will come into play and local clustering of particles will occur, forming quasi-bound states. The energy of the system will be decreased by the amount of the binding energy. One can thus increase the mass limit for neutron stars by letting local clustering of particles take place.

If one piles all the mass of the universe into a neutron star and considers Wheeler's static problem, then essentially, if one still wants to save the baryon conservation law, the binding energy of these clusters must be of the order of the rest energy of the particles themselves. Such a *super-strong field has not been observed* in present high energy physics experiments.

If the fine structure constant is of order unity, a number of interesting things will happen. Pairs of electrons will be created spontaneously into bound states; yet these bound pairs have zero energy. In order that such thing be possible for baryons, the strength of the coupling $f^2/\hbar c$ must be at least 100 times stronger than that for the strong interaction. In order not to conflict with the present knowledge of elementary particles, the range of this super-strong interaction cannot exceed, say, 10^{-15} cm. Since a binding among all pairs is assumed (attractive force), the interactions must be scalar or tensor in character. The quanta associated with such a field must be around 100 times the π meson mass, or 20 times the nucleon mass.

Of course, what we have suggested is purely speculative. A star which collapses to within its Schwarzschild singularity will gradually become invisible. On the other hand, if a star can get rid of its mass-energy by resorting to baryon non-conservation or the existence of such a hypothetical

*Private communication. The author first learned this idea from Professor C. W. Misner, who believed he was quoting Dr. Weinberg. When the author asked Dr. Weinberg, he insisted that he was misquoted by one of us (H. Y. C. or C. W. M.). Nevertheless, he admitted later that this idea might work. His original idea applied to model universes as a whole, but he thinks this idea may work inside a neutron star too. The author feels that, although the present idea originates from misquotations, Dr. Weinberg nevertheless should deserve the credit of being the originator.

super-strong interaction, the remaining body should be observable with approximately the surface characteristic discussed in Section 8. They are certainly observable.

10. CONCLUDING REMARKS*

In this paper the final evolution of a pre-supernova star is reviewed. It seems inevitable that the star will end up as a neutron star with a radius of around 10^6 cm and a surface temperature of around 10^6 - 10^7°K. A critical mass limit exists for neutron stars. Beyond this mass limit no static structure is possible. Neutron stars can only be detected by extra-terrestrial x-ray telescopes. If detected, they pose interesting questions on our present theory of fundamental particles.

11. ACKNOWLEDGMENTS

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* When this article went to press, rocketbound x-ray observations made by Rossi's group (H. Gursky, R. Giacconi, F. R. Paolini, and B. B. Rossi, *Phys. Rev. Letters*, 11: 530 (1963)) and H. Friedman's group (reported in the December meeting of the American Astronomical Society, 1963) reveal the existence of discrete x-ray sources, one of them near the Crab Nebula, the remnant of the supernova of 1054 A.D. The energy flux in the 3A wavelength region is around 10^{-7} ergs/cm^2-sec. If the source is indeed neutron stars in the Crab Nebula, the calculated source flux is around 10^{37} ergs/sec. This is consistent with the estimates made in this paper. However, further observations are necessary to clarify if these x-ray sources are indeed neutron stars.

A more detailed evaluation of the surface properties of neutron stars has been performed by the author and also by Miss S. Tsuruta and Dr. A.G.W. Cameron. Both works are to be published elsewhere. Results indicated that the crude estimates made in this paper are correct to one order of magnitude.

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Appendix A
Virial Theorem

Now we derive the virial theorem. Defining $V \equiv (4\pi/3)r^3$ and multiplying both sides of Equation 2.9 by V, we have

$$V\,dP = -\rho\frac{Gm}{r^2}\cdot\frac{4\pi}{3}r^3\,dr = -\frac{1}{3}\frac{Gm\,dm}{r} . \tag{A1}$$

We now integrate Equation (A1) over the star. Partially integrating $\int V\,dP$, we have

$$\int V\,dP = PV\Big|_{r=0}^{r=R} - \int P\,dV = -\int P\,dV \tag{A2}$$

since, at $r = 0$, $V = 0$; and, at $r = R$ (radius of the star), $P = 0$. Thus, we have the relation

$$3\int P\,dV = \int\frac{Gm\,dm}{r} = -E_G . \tag{A3}$$

If a relation between P and E (energy density) exists, Equation (A3) relates the gravitational energy of a star to its thermodynamic energy. For a non-relativistic gas $P = 2/3E$; and for a relativistic gas, $P = 1/3E$. Thus the total energy E_T of a star is

$$E_T = -\eta E_{th} , \tag{A4}$$

where $0 \le \eta \le 1$.

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